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Numerical Analysis

On Zienkiewicz–Zhu error estimators for Maxwell’s equations

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Abstract

We consider a posteriori Zienkiewicz–Zhu (ZZ) type error estimators for the Maxwell equations. The main tool is the use of appropriate recovered values of the electric field and its curl. *To cite this article: S. Nicaise, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Résumé

Estimateurs d’erreur du type Zienkiewicz–Zhu pour les équations de Maxwell. Nous considérons des estimateurs d’erreur a posteriori du type Zienkiewicz–Zhu (ZZ) pour les équations de Maxwell. L’ingrédient principal est d’utiliser des valeurs nodales reconstituées du champ électrique et de son rotationnel. *Pour citer cet article : S. Nicaise, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Version française abrégée

Introduction

Dans cette Note, nous considérons les équations de Maxwell (1) dans un domaine borné Ω de l’espace, où \mathbf{u} correspond au champ électrique et $\mathbf{f} \in L^2(\Omega)^3$ est à divergence nulle. Les coefficients χ et β sont ici supposés constants et positifs dans tout Ω .

Le problème (1), ou plutôt sa formulation variationnelle (2), est approché par l’espace des éléments finis d’arêtes de Nédélec de plus petit degré V_h défini par (3) (cf. [6]) et basée sur une famille régulière (au sens de Ciarlet cf. [3]) de triangulations \mathcal{T}_h du domaine faites de tétraèdres. Notons \mathbf{u}_h la solution du problème approché (4). Un estimateur d’erreur a posteriori du type résiduel efficace et robuste pour la norme $H_0(\text{curl}, \Omega)$ de $\mathbf{u} - \mathbf{u}_h$ a été analysé dans [1]

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(voir aussi [5,7]). Mais à notre connaissance, aucun estimateur du type Zienkiewicz–Zhu n’existe pour le système de Maxwell. Notre but est donc d’introduire un tel estimateur et de montrer son équivalence avec l’estimateur du type résiduel mentionné ci-dessus. Notre approche est une adaptation de l’approche de [8] et utilise de manière essentielle l’introduction de valeurs nodales reconstituées pour des champs de vecteurs \mathbb{P}^1 par morceaux, similaires à des gradients reconstitués.

Estimateurs d’erreur

Les estimateurs de type résiduel standards et nodaux étant définis par les Définitions 3.1 et 3.2, nous montrons l’équivalence (3.3) entre les estimateurs nodaux grâce aux propriétés de l’espace V_h .

L’estimateur du type Zienkiewicz–Zhu est introduit à la Définition 3.7 et utilise les valeurs nodales reconstituées de la Définition 3.5, valables pour n’importe quel champ de vecteurs \mathbb{P}^1 par morceaux.

Nous montrons ensuite les équivalences suivantes :

Théorème 0.1. *Pour tout noeud x ou triangle T de la triangulation \mathcal{T}_h , les estimées (13) à (15) sont satisfaites.*

La preuve de ces équivalences est basée sur l’équivalence (3.3) et le Lemme 3.6 (qui découle du Lemme 3.12 of [4]).

Ce résultat et l’équivalence entre l’erreur et l’estimateur du type résiduel (cf. Théorème 3.4) permettent d’obtenir le

Théorème 0.2. *L’erreur est bornée localement supérieurement et globalement inférieurement comme suit :*

$$\eta_{Z,T} \lesssim \| \mathbf{u} - \mathbf{u}_h \|_{H(\text{curl}, \omega_T)} + \sum_{T' \subset \omega_T} \zeta_{T'}, \quad \forall T \in \mathcal{T}_h,$$

$$\| \mathbf{u} - \mathbf{u}_h \|_{H(\text{curl}, \Omega)} \lesssim \eta_Z + \eta_{el} + \zeta.$$

1. Introduction

Using a time discretization of the classical eddy current electric formulation in a three-dimensional bounded polyhedral domain Ω we have to solve at each timestep the Maxwell equation [2,1]

$$\begin{cases} \text{curl}(\chi \text{curl} \mathbf{u}) + \beta \mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where \mathbf{u} is the time approximation of the electric field \mathbf{E} and \mathbf{f} is in $L^2(\Omega)^3$ and divergence free. The coefficient χ is the inverse of the magnetic permeability, and β is a multiple of σ , the conductivity of the body occupying Ω . Here for the sake of simplicity, we assume that χ and β are constant and positive in the whole of Ω .

The variational formulation of this problem is well known and requires the introduction of the space

$$H_0(\text{curl}, \Omega) = \{ \mathbf{u} \in L^2(\Omega)^3 : \text{curl} \mathbf{u} \in [L^2(\Omega)]^3 \text{ and } \mathbf{u} \times \mathbf{n} = 0 \text{ on } \Gamma \},$$

and of the bilinear form

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\chi \text{curl} \mathbf{u} \cdot \text{curl} \mathbf{v} + \beta \mathbf{u} \cdot \mathbf{v}) \, dx.$$

The weak formulation of (1) consists in finding $\mathbf{u} \in H_0(\text{curl}, \Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in H_0(\text{curl}, \Omega), \tag{2}$$

where (\cdot, \cdot) is the $[L^2(\Omega)]^3$ -inner product. Since a is clearly a coercive sesquilinear form on $H_0(\text{curl}, \Omega)$, problem (2) has a unique solution by the Lax–Milgram lemma.

Problem (2) is approximated in the lower order Nédélec edge finite element space V_h based on a triangulation \mathcal{T}_h of the domain made of isotropic tetrahedra (cf. [6]). If \mathbf{u}_h is the solution of the discretization of (2) an efficient and reliable residual a posteriori error estimator for the error $\mathbf{u} - \mathbf{u}_h$ in the $H_0(\text{curl}, \Omega)$ -norm was investigated in [1] (see also [5,7]). But to our knowledge, no a posteriori Zienkiewicz–Zhu error estimator exists for Maxwell’s system. The goal of this Note is then to introduce such a Zienkiewicz–Zhu error estimator and show its equivalence with the residual error estimator. This is made by adapting the approach of [8] (see also [9, Section 1.5]) and by introducing an appropriate recovered value for piecewise \mathbb{P}^1 vector valued functions on the triangulation, similar to the recovered gradient.

Let us finish this introduction with some notation used in the whole paper: For shortness the $L^2(D)$ -norm will be denoted by $\|\cdot\|_D$. In the case $D = \Omega$, we will drop the index Ω . The standard $H(\text{curl}, D)$ -norm is denoted by $\|\cdot\|_{H(\text{curl}, D)} = \|\cdot\|_D + \|\text{curl} \cdot\|_D$. Finally, the notation $a \lesssim b$ and $a \sim b$ means the existence of positive constants C_1 and C_2 (which are independent of \mathcal{T}_h and of the function under consideration) such that $a \leq C_2 b$ and $C_1 b \leq a \leq C_2 b$, respectively.

2. Discretization of the Maxwell equations

The domain Ω is discretized by a family of conforming meshes \mathcal{T}_h , $h > 0$ made of tetrahedra and regular in Ciarlet’s sense cf. [3].

Elements will be denoted by T or T' , its faces are denoted by E and its nodes by x . The set of (internal or boundary) faces of the triangulation will be denoted by \mathcal{E} , while \mathcal{N} will be the set of all (internal or boundary) nodes of the mesh.

For a face E of an element T introduce the *outer normal vector* by $\mathbf{n} = (n_1, n_2, n_3)^\top$. Furthermore, for each face E we fix one of the two normal vectors and denote it by \mathbf{n}_E .

The *jump* of some (scalar or vector valued) function v across a face E is then defined as

$$[[v(y)]] := \lim_{\alpha \rightarrow +0} v(y + \alpha \mathbf{n}_E) - v(y - \alpha \mathbf{n}_E), \quad y \in E.$$

Furthermore one requires local subdomains (also known as patches). As usual, let ω_T be the union of all elements having a common face with T . By ω_x we denote the union of all elements having x as node.

We denote by V_h the lowest order Nédélec finite element space

$$V_h := \{ \mathbf{v}_h \in H_0(\text{curl}, \Omega) : \mathbf{v}_h|_K \in \mathcal{ND}_1, \forall K \in \mathcal{T}_h \}, \quad (3)$$

where the set \mathcal{ND}_1 is defined by $\mathcal{ND}_1 = \{ p(\mathbf{x}) = a + b \times \mathbf{x} : a, b \in \mathbb{R}^3 \}$, when $\mathbf{x} = (x_1, x_2, x_3)^\top$ (see [6]).

As $V_h \subset H_0(\text{curl}, \Omega)$ the discrete problem associated with (2) is to find $\mathbf{u}_h \in V_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h. \quad (4)$$

3. Error estimators

3.1. Residual error estimators

The exact residuals are defined by $R_T := \mathbf{f} - (\chi \text{curl curl } \mathbf{u}_h + \beta \mathbf{u}_h)$ on T , its approximated residuals will be denoted by r_T .

Definition 3.1. The local and global residual error estimators are defined by

$$\eta_T^2 := h_T^2 \|r_T\|_T^2 + h_T \sum_{E \subset \partial T \setminus \Gamma} (\|\mathbf{u}_h \cdot \mathbf{n}_E\|_E^2 + \|\operatorname{curl} \mathbf{u}_h \times \mathbf{n}_E\|_E^2), \quad \eta^2 := \sum_{T \in \mathcal{T}_h} \eta_T^2, \tag{5}$$

$$\zeta_T^2 := h_T^2 \|r_T - R_T\|_T^2, \quad \zeta^2 := \sum_{T \in \mathcal{T}_h} \zeta_T^2. \tag{6}$$

For our further analysis, we introduce node related quantities.

Definition 3.2. The local and global nodal estimators are defined by

$$\eta_{R,x}^2 := h_x \sum_{E \subset \Omega: x \in \bar{E}} (\|\mathbf{u}_h \cdot \mathbf{n}_E\|_E^2 + \|\operatorname{curl} \mathbf{u}_h \times \mathbf{n}_E\|_E^2), \quad \eta_R^2 := \sum_{x \in \mathcal{N}} \eta_{R,x}^2, \tag{7}$$

$$\eta_{\check{R},x}^2 = h_x \sum_{E \subset \Omega: x \in \bar{E}} (\|\llbracket \mathbf{u}_h \rrbracket\|_E^2 + \|\llbracket \operatorname{curl} \mathbf{u}_h \rrbracket\|_E^2). \tag{8}$$

We first remark that

$$\eta \sim \eta_R + \eta_{\text{el}},$$

where η_{el} corresponds to the element residuals, i.e.,

$$\eta_{\text{el}}^2 = \sum_{T \in \mathcal{T}_h} h_T^2 \|r_T\|_T^2.$$

Moreover the next equivalence between $\eta_{R,x}$ and $\eta_{\check{R},x}$ is essential for our further analysis.

Lemma 3.3. *The following equivalence holds:*

$$\eta_{R,x} \sim \eta_{\check{R},x}. \tag{9}$$

Proof. It suffices to remark that the inclusion $\mathbf{u}_h \in V_h \subset \mathcal{H}_0(\operatorname{curl}; \Omega)$ implies that

$$\llbracket \mathbf{u}_h \times \mathbf{n}_E \rrbracket = 0 \quad \text{on } E, \forall E \in \mathcal{E}, E \subset \Omega.$$

As a consequence we may write

$$\llbracket \mathbf{u}_h \cdot \mathbf{n}_E \rrbracket = \pm \llbracket \mathbf{u}_h \rrbracket \quad \text{on } E \in \mathcal{E}, E \subset \Omega.$$

Similarly as $\operatorname{curl} \mathbf{u}_h$ belongs to $L^2(\Omega)^3$, $\operatorname{div} \operatorname{curl} \mathbf{u}_h = 0$ in Ω and therefore

$$\llbracket \operatorname{curl} \mathbf{u}_h \cdot \mathbf{n}_E \rrbracket = 0 \quad \text{on } E, \forall E \in \mathcal{E}, E \subset \Omega.$$

As before we deduce that

$$\llbracket \operatorname{curl} \mathbf{u}_h \times \mathbf{n}_E \rrbracket = \pm \llbracket \operatorname{curl} \mathbf{u}_h \rrbracket \quad \text{on } E \in \mathcal{E}, E \subset \Omega.$$

These two identities prove the assertion. \square

To finish this subsection, let us recall the following residual error estimate proved in Theorem 4 of [1] (see also Corollaries 4.4 and 4.7 of [7]).

Theorem 3.4. *The error $\mathbf{u} - \mathbf{u}_h$ is bounded locally from below and globally from above as follows:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{curl}, \Omega)} \lesssim \eta + \zeta, \tag{10}$$

$$\eta_T \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{curl}, \omega_T)} + \sum_{T' \subset \omega_T} \zeta_{T'}, \quad \forall T \in \mathcal{T}_h. \tag{11}$$

3.2. The ZZ error estimator

Inspired from [4], for a vector valued function \mathbf{v}_h in W_h , the space of piecewise linear vector fields on the triangulation, we define its recovered value in $V_h := W_h \cap C(\Omega, \mathbb{R}^d)$. First recall that for any node $x \in \mathcal{N}$, λ_x is the standard scalar valued hat function associated with x , namely, λ_x is a continuous piecewise linear function on the triangulation satisfying

$$\lambda_x(y) = \delta_{x,y}, \quad \forall y \in \mathcal{N}.$$

Definition 3.5. The arbitrary recovered operator R is defined by $R : W_h \rightarrow V_h : \mathbf{v}_h \rightarrow \sum_{x \in \mathcal{N}} (R\mathbf{v}_h)(x)\lambda_x$, where the nodal values are given by

$$(R\mathbf{v}_h)(x) := \sum_{T \subset \omega_x} \mu_{T,x} \mathbf{v}_{h|T}(x), \quad \forall x \in \mathcal{N}, \tag{12}$$

where the weights $\mu_{T,x} \geq 0$ can be chosen arbitrarily such that $\sum_{T \subset \omega_x} \mu_{T,x} = 1$.

Note that the difference with the so-called recovered gradient is that \mathbf{v}_h is not necessarily piecewise constant. Note further that a standard choice is to take the weight $\mu_{T,x} := |T|/|\omega_x|$, for any $T \subset \omega_x$. In that case, if \mathbf{v}_h is piecewise constant, then $R\mathbf{v}_h$ is the orthogonal projection of \mathbf{v}_h by means of a projection with respect to a particular scalar product [10,9].

The main property of the recovered value is given in the next Lemma, consequence of Lemma 3.12 of [4]:

Lemma 3.6. Consider an arbitrary node x and the associated patch ω_x . Let \mathbf{v}_h be a vector valued function in W_h , then the next equivalence holds

$$\sum_{E \subset \Omega : x \in \bar{E}} \|\llbracket \mathbf{v}_h(x) \rrbracket\|^2 \sim \sum_{T \subset \omega_x} |R\mathbf{v}_h(x) - \mathbf{v}_{h|T}(x)|^2.$$

The corresponding ZZ estimator is given next.

Definition 3.7. The local and global ZZ estimators are given by

$$\eta_{Z,T}^2 := \|R\mathbf{u}_h - \mathbf{u}_h\|_T^2 + \|R(\text{curl}_h \mathbf{u}_h) - \text{curl}_h \mathbf{u}_h\|_T^2, \quad \eta_Z^2 := \sum_{T \in \mathcal{T}_h} \eta_{Z,T}^2.$$

Now we are able to prove the equivalence between the residual and the ZZ estimators.

Theorem 3.8. The following local and global relations hold (for all $x \in \mathcal{N}$ or $T \in \mathcal{T}_h$).

$$\eta_{R,x}^2 \lesssim \sum_{T \subset \tilde{\omega}_x} \eta_{Z,T}^2, \tag{13}$$

$$\eta_{Z,T}^2 \lesssim \sum_{x \in \mathcal{N} : x \in \bar{T}} \eta_{R,x}^2, \tag{14}$$

$$\eta_R \sim \eta_Z, \tag{15}$$

where $\tilde{\omega}_x = \bigcup_{E \subset \Omega : x \in \bar{E}} \cup_{y \in \bar{E}} \omega_y$.

Proof. The equivalence (15) is a direct consequence of (13) and (14).

By Lemma 3.3, a scaling argument and the fact that all norms are equivalent in a finite dimensional space, we may write

$$\begin{aligned} \eta_{R,x}^2 &\sim h_x \sum_{E \subset \Omega: x \in \bar{E}} |E| \left(\sum_{y \in \bar{E}} \|\llbracket \mathbf{u}_h(y) \rrbracket\|_E^2 + \|\llbracket \operatorname{curl} \mathbf{u}_h(x) \rrbracket\|_E^2 \right) \\ &\sim h_x^3 \sum_{E \subset \Omega: x \in \bar{E}} \left(\sum_{y \in \bar{E}} \|\llbracket \mathbf{u}_h(y) \rrbracket\|_E^2 + \|\llbracket \operatorname{curl} \mathbf{u}_h(x) \rrbracket\|_E^2 \right). \end{aligned}$$

Now making use of Lemma 3.6, we obtain

$$\eta_{R,x}^2 \sim h_x^3 \sum_{E \subset \Omega: x \in \bar{E}} \sum_{y \in \bar{E}} \sum_{T \subset \omega_y} |R\mathbf{u}_h(y) - \mathbf{u}_h|_T(y)|^2 + h_x^3 \sum_{T \subset \omega_x} |R(\operatorname{curl}_h \mathbf{u}_h)(x) - (\operatorname{curl}_h \mathbf{u}_h)|_T(x)|^2.$$

By finite dimensionality (see above), we conclude that

$$\eta_{R,x}^2 \lesssim \sum_{E \subset \Omega: x \in \bar{E}} \sum_{y \in \bar{E}} \sum_{T \subset \omega_y} \|R\mathbf{u}_h - \mathbf{u}_h|_T\|_T^2 + \sum_{T \subset \omega_x} \|R(\operatorname{curl}_h \mathbf{u}_h) - (\operatorname{curl}_h \mathbf{u}_h)|_T\|_T^2.$$

This proves (13). The converse relation (14) is proved similarly. \square

Theorem 3.9. *The error is bounded locally from below and globally from above:*

$$\eta_{Z,T} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{curl}, \omega_T)} + \sum_{T' \subset \omega_T} \zeta_{T'}, \quad \forall T \in \mathcal{T}_h, \quad (16)$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{curl}, \Omega)} \lesssim \eta_Z + \eta_{\text{el}} + \zeta. \quad (17)$$

Proof. These are immediate consequences of Theorems 3.4 and 3.8. \square

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