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Partial Differential Equations

Existence via compactness for maximal monotone elliptic operators

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Abstract

In this Note we propose a new method of proving the existence of solutions to $-\operatorname{div} A(x, \nabla u) \ni f$, when $A(x, \nabla u)$ has x-dependent maximal monotone graph. The idea is based on the theory of Young measures and on the method of compensated compactness. Alternative approaches were proposed elsewhere. However, our method allows us to obtain also the strong convergence of approximate solutions. ***To cite this article: P. Gwiazda, A. Zatorska-Goldstein, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Existence par compacticité pour opérateurs maximaux monotone elliptiques. Dans cette Note nous proposons une méthode nouvelle de démonstation de l'existence de solutions de $-\operatorname{div} A(x, \nabla u) \ni f$, où $A(x, \nabla u)$ a un graphe maximale monotone dépendant de x. L'idée de cette méthode est d'utiliser la théorie des mésures de Young et la méthode de compacticité par compensation. Une autre approche a été proposée ailleurs. Néanmoins, notre méthode permet d'obtenir la convergence forte des solutions approchées. ***Pour citer cet article : P. Gwiazda, A. Zatorska-Goldstein, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction and statement of the results

Let Ω be an open bounded subset of $\mathbb{R}^m$. Given a function $A = A(x, \xi) : \Omega \times \mathbb{R}^m \to 2^{\mathbb{R}^m}$, we consider the following elliptic differential inclusion in divergence form $-\operatorname{div} A(x, \nabla u) \ni f$ for the unknown function $u : \Omega \to \mathbb{R}$. In the paper by Chiadò Piat, Dal Maso, Defrancheschi [4] a set of assumptions on A was stated, and the first proof for such situations was achieved. The crucial point was defining the proper measurability of A with respect to x. Note that if A is multi-valued, there are many possible choices.

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Recently an entirely new approach was proposed by Francfort, Murat and Tartar [5]. They reformulate the assumptions into a completely equivalent form, but omitting the use of multi-valued techniques. Our aim is to propose a different method of proving the existence of solutions, with the same assumptions as in [4,5], replacing only monotonicity by a strict monotonicity. It yields additional information about the strong convergence of the approximate solutions. Contrary to the other two papers, our method follows the spirit of the compactness method of J.-L. Lions for variational-type operators (see [6], Chapter 2.6; Theorem 2.8 and [3], Lemma 5); however, we use Young measures and compensated compactness in a non standard setting. Let us now state the main results.

Proposition 1.1. *Assume that* $A = A(x,\xi) : \Omega \times \mathbb{R}^m \to 2^{\mathbb{R}^m}$ *is maximal strictly monotone in* ξ *for a.e.* $x \in \Omega$ *and* $\mathcal{A}(x) \subset \mathbb{R}^m \times \mathbb{R}^m$ *is an* x*-dependent graph of* $A(x,\cdot)$ *for a.e.* $x \in \Omega$*. Moreover, assume that* $\mathcal{A}(x)$ *has following properties:*

(i) *There exist* $1 < p < \infty$, $m(x) \geqslant 0$ *in* $L^1(\Omega)$ *and* $\alpha > 0$ *such that for a.e.* x *in* Ω *and every* $(e,d) \in \mathcal{A}(x)$,

$$-\langle d|e\rangle \leqslant m(x) - \alpha\big(|e|^p + |d|^{p'}\big).$$

(ii) *For any closed subset* C *of* $\mathbb{R}^m$ *the set* $\{(x,e) \in \Omega \times \mathbb{R}^m\colon$ *there exists* $d \in C$ *such that* $(e,d) \in \mathcal{A}(x)\}$ *is measurable with respect to the* σ*-field generated by* $\mathcal{L}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m)$.

Then for every $f \in W^{-1,p'}(\Omega)$ *there exists a pair* (σ,u) *such that* $\sigma\colon\ \Omega \to \mathbb{R}^m$ *is measurable,* $u \in W_0^{1,p}(\Omega)$, $(\sigma(x), \nabla u(x)) \in \mathcal{A}(x)$ *for a.e.* $x \in \Omega$*, and* $-\operatorname{div}\sigma = f$ *in* $\mathcal{D}'(\Omega)$.

Above and in the following, $\langle\cdot\|\cdot\rangle$ denotes the scalar product in $\mathbb{R}^m$. For brevity in this Note we prove the easier case – when the graph $\mathcal{A}$ does not depend on x and we add a comment on general case.

2. The compactness method

The next theorem is a modification of the fundamental theorem on Young measures. We replace the families of single distributed probabilistic measures (compare [1]) by general probabilistic measures, obtaining:

Theorem 2.1. *Let* Ω *be an open bounded subset of* $\mathbb{R}^m$*. Assume that for every* $x \in \Omega$ *there exists a sequence of probability measures* ν_x^j *on* $\mathbb{R}^N$ *such that for every* j*, the mapping* $\nu^j : \Omega \to \mathcal{M}(\mathbb{R}^N)$ *is weak-*$*$ *measurable. Assume* $\nu : \Omega \to \mathcal{M}(\mathbb{R}^N)$ *to be such that* $\nu^j \overset{*}{\rightharpoonup} \nu$ *in* $L_w^\infty(\Omega, \mathcal{M}(\mathbb{R}^N))$.

If the sequence ν^j *satisfies the 'tightness condition',*

$$\lim_{M\to\infty} \sup_j \big|\big\{x \in \Omega\colon\ \operatorname{supp}(\nu_x^j) \setminus B(0,M) \neq \emptyset\big\}\big| \to 0, \tag{1}$$

then:

(i) $|\nu_x|_{\mathcal{M}(\mathbb{R}^N)} = 1$ *a.e. in* Ω;

(ii) *for every measurable subset* $E \subset \Omega$ *and for every Carathéodory function* f *such that*

$$\lim_{R\to\infty} \sup_{j\in\mathbb{N}} \int_E \int_{\{\lambda\in\mathbb{R}^N\colon\ |f(x,\lambda)|>R\}} \big|f(x,\lambda)\big|\, \mathrm{d}\nu_x^j(\lambda)\, \mathrm{d}x = 0, \tag{2}$$

we have:

$$\int_{\mathbb{R}^N} f(x,\lambda)\, \mathrm{d}\nu_x^j(\lambda) \rightharpoonup \int_{\mathbb{R}^N} f(x,\lambda)\, \mathrm{d}\nu_x(\lambda) \quad \text{in } L^1(E).$$

Throughout this Note by ϕ^ε we will denote the function $\phi^\varepsilon(\xi) = \frac{1}{\varepsilon^m}\phi(\frac{\xi}{\varepsilon})$, where $\varepsilon > 0$ and $\phi \in C_0^\infty$ is a nonnegative function such that $\int_{\mathbb{R}^m} \phi(\xi)\,\mathrm{d}\xi = 1$.

Proof of Proposition 1.1 in the case A independent on x. Let us first observe that if A satisfies the assumptions of Proposition 1.1 and does not depend on x, then there exists a selection a from A such that: a is Borel measurable, $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{R}^m)$ and is strictly monotone, i.e. for every $\xi_1, \xi_2 \in \mathbb{R}^m$, $\xi_1 \neq \xi_2$

$$\big\langle a(\xi_1) - a(\xi_2) \mid \xi_1 - \xi_2\big\rangle > 0. \tag{3}$$

Moreover, for all $\xi \in \mathbb{R}^m$, the following growth and coercivity conditions are satisfied:

$$\big|a(\xi)\big| \leqslant c_1\big(1 + |\xi|^{p-1}\big), \qquad \big\langle a(\xi) \mid \xi\big\rangle \geqslant c_2|\xi|^p - c_3, \tag{4}$$

where c_1, c_2, c_3 are strictly positive. Define a function $a^\varepsilon(\xi) = (a * \phi^\varepsilon)(\xi)$. The regularization preserves the monotonicity condition (3). The growth and coercivity conditions (4) are preserved up to a possible choice of the new constants c_1', c_2' and c_3' independent of ε for $|\varepsilon| \leqslant 1$. Thus, one can show that there exists a weak solution u^ε to the problem $-\operatorname{div} a^\varepsilon(\nabla u^\varepsilon) = f$, $u^\varepsilon|_{\partial\Omega} = 0$. The energy estimates and the conditions imposed on a yield also a uniform bound on the $W_0^{1,p}$ norm of solutions. Therefore, up to subsequences, it holds: $\nabla u^\varepsilon \rightharpoonup \nabla u$ in $L^p(\Omega, \mathbb{R}^m)$, $a^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \sigma$ in $L^{p'}(\Omega, \mathbb{R}^m)$, where σ is a measurable function. The div–curl lemma of the theory of compensated compactness provides

$$\big\langle a^\varepsilon(\nabla u^\varepsilon) \mid \nabla u^\varepsilon\big\rangle \to \langle \sigma \mid \nabla u\rangle \quad \text{in } \mathcal{D}'(\Omega). \tag{5}$$

We have:

$$a^\varepsilon\big(\nabla u^\varepsilon(x)\big) = \int_{\mathbb{R}^m} a(\xi)\phi^\varepsilon\big(\nabla u^\varepsilon(x) - \xi\big)\,\mathrm{d}\xi = \int_{\mathbb{R}^m} a(\xi)\,\mathrm{d}\mu_x^\varepsilon(\xi),$$

where μ_x^ε is a nonnegative probability measure, absolutely continuous with respect to the Lebesgue measure, with density $\phi^\varepsilon(\nabla u^\varepsilon(x) - (\cdot))$. Define a function $g : \mathbb{R}^m \to \mathbb{R}^m$ by $g(\xi) = a(\xi) + \xi$. The monotonicity condition (3) implies that g is injective and the function $g^{-1} : \operatorname{im} g \to \mathbb{R}^m$ is Lipschitz continuous. Moreover $a(g^{-1}(\cdot))$ is continuous on $\operatorname{im} g$. Define a measure $\nu_x^\varepsilon \in \mathcal{M}(\operatorname{im} g)$ by:

$$\nu_x^\varepsilon(S) = \mu_x^\varepsilon\big(g^{-1}(S)\big) \quad \text{for every Borel set } S \subset \operatorname{im} g. \tag{6}$$

For every ε the measure ν_x^ε is a probability measure on $\operatorname{im} g$ and the mapping $\nu^\varepsilon : \Omega \to \mathcal{M}(\operatorname{im} g)$ is weak-$*$ measurable, as it follows from the measurability conditions of g. The Banach–Alaoglu Theorem yields that there exists a weak-$*$ measurable mapping $\nu \in L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^m))$ such that $\nu^\varepsilon \overset{*}{\rightharpoonup} \nu$ and $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} \leqslant 1$. Then testing the weak-$*$ convergence with a suitable test function, we obtain:

$$\operatorname{supp} \nu_x \subset K, \quad \text{where } K = \overline{\operatorname{im} g}.$$

Note that $|\gamma^\varepsilon|_{L^1} \leqslant C$, where $\gamma^\varepsilon(x) := \max_{\xi \in \operatorname{supp}(\mu_x^\varepsilon)} |g(\xi)| = \max_{\lambda \in \operatorname{supp} \nu_x^\varepsilon} |\lambda|$. This implies that the sequence of mappings ν^ε satisfies the tightness condition (1). Then, Theorem 2.1 implies that the measure ν_x is a probability measure on K for a.e. $x \in \Omega$. It is easy to check that

$$\nabla u^\varepsilon(x) = \int_{\mathbb{R}^m} \xi\,\mathrm{d}\mu_x^\varepsilon(\xi) + \mathrm{O}(\varepsilon) \quad \text{and} \quad \big\langle a^\varepsilon\big(\nabla u^\varepsilon(x)\big) \bigm| \nabla u^\varepsilon(x)\big\rangle = \int_{\mathbb{R}^m} \big\langle a(\xi) \mid \xi\big\rangle d\mu_x^\varepsilon(\xi) + \mathrm{O}(\varepsilon).$$

Therefore we get:

$$\begin{aligned} &a^\varepsilon\big(\nabla u^\varepsilon(x)\big) = \int_{\operatorname{im} g} a\big(g^{-1}(\lambda)\big)\,\mathrm{d}\nu_x^\varepsilon(\lambda), \qquad \nabla u^\varepsilon(x) = \int_{\operatorname{im} g} g^{-1}(\lambda)\,\mathrm{d}\nu_x^\varepsilon(\lambda) + \mathrm{O}(\varepsilon), \\ &\big\langle a^\varepsilon\big(\nabla u^\varepsilon(x)\big) \bigm| \nabla u^\varepsilon(x)\big\rangle = \int_{\operatorname{im} g} \big\langle a\big(g^{-1}(\lambda)\big) \bigm| g^{-1}(\lambda)\big\rangle\,\mathrm{d}\nu_x^\varepsilon(\lambda) + \mathrm{O}(\varepsilon). \end{aligned} \tag{7}$$

We may interpret the measures ν_x^ε as measures defined on $\mathbb{R}^m$. The functions g^{-1} and $a(g^{-1})$ can be continuously extended onto K. We will denote these extensions by $\tilde{g}^{-1}$ and $\tilde{a}$. It follows from (7), (5) and the second part of Theorem 2.1, that

$$\begin{aligned}&\sigma(x) = \int_K \tilde{a}(\lambda)\,\mathrm{d}\nu_x(\lambda), \qquad \nabla u(x) = \int_K \tilde{g}^{-1}(\lambda)\,\mathrm{d}\nu_x(\lambda),\\ &\big\langle \sigma(x) \mid \nabla u(x)\big\rangle = \int_K \big\langle \tilde{a}(\lambda) \mid \tilde{g}^{-1}(\lambda)\big\rangle \mathrm{d}\nu_x(\lambda) + \beta,\end{aligned} \tag{8}$$

where β is a nonnegative Radon measure describing concentrations, cf. [2].

Maximality of $\mathcal{A}$ implies that $(\tilde{g}^{-1}(\lambda), \tilde{a}(\lambda)) \in \mathcal{A}$. Moreover, the strict monotonicity of the graph together with (8) ensure that $\int_K \langle \tilde{a}(\lambda) - a(\nabla u(x)) \mid \tilde{g}^{-1}(\lambda) - \nabla u(x)\rangle\,\mathrm{d}\nu_x(\lambda) = 0$ which implies $\tilde{g}^{-1}(\lambda) = \nabla u(x)$ for ν_x-a.e. λ and therefore $(\nabla u(x), \tilde{a}(\lambda)) \in \mathcal{A}$ for ν_x-a.e. λ. Since maximal monotone operators are convex-valued, hence $(\nabla u(x), \sigma(x)) \in \mathcal{A}$, where σ is given by (8). Moreover, μ_x (the weak-$*$ limit of the sequence of the measures μ_x^ε) is a Dirac measure on $\mathbb{R}^m$ for a.e. x in Ω. This is provided by $\{\nabla u(x)\} = \tilde{g}^{-1}(\operatorname{supp}\nu_x) = \operatorname{supp}\mu_x$ for a.a. $x \in \Omega$. It follows that $\nabla u^\varepsilon \to \nabla u$ a.e. in Ω. The proof is complete. □

Idea of the proof in the case $A = A(x, \nabla u)$. To avoid problems with measurability with respect to x we use the method introduced in [5]. There exists a Carathéodory function $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ such that $\varphi(x, \cdot)$ is a contraction for a.e. x in Ω and

$$(\xi, \eta) \in \mathcal{A}(x) \quad \Longleftrightarrow \quad \eta - \xi = \varphi(x, \eta + \xi). \tag{9}$$

Fix $x \in \Omega$ and define functions:

$$\tilde{a}_x(\lambda) = \frac{1}{2}\big(\lambda + \varphi(x, \lambda)\big), \qquad \tilde{g}_x^{-1}(\lambda) = \frac{1}{2}\big(\lambda - \varphi(x, \lambda)\big). \tag{10}$$

These are continuous functions on $\mathbb{R}^m$ and for every $\lambda \in \mathbb{R}^m$ the mappings $x \mapsto \tilde{a}_x(\lambda)$ and $x \mapsto \tilde{g}_x^{-1}(\lambda)$ are measurable. Moreover, for a.e. x in Ω and for every $\lambda \in K_x$: $\tilde{a}_x(\lambda) = a(x, g_x^{-1}(\lambda))$ and $\tilde{g}_x^{-1}(\lambda) = g_x^{-1}(\lambda)$ and therefore for every $\lambda \in K_x$ we have $(\tilde{g}_x^{-1}(\lambda), \tilde{a}_x(\lambda)) \in \mathcal{A}(x)$.

As in the previous case we regularize the function a by a convolution in ξ. The measurability and continuity of the functions $\tilde{a}_x$ and $\tilde{g}_x^{-1}$ allow us to avoid problems caused by the fact that the measures ν_x have supports dependent on x. □

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