Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 340 (2005) 287–290

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Differential Geometry

Uniqueness of extremal Kähler metrics

Xiuxiong Chen, Gang Tian [1]

Department of Mathematics, University of Wisconsin, Madison, WI 53706-1, USA

Received 4 November 2004; accepted 5 November 2004

Available online 22 January 2005

Presented by Jean-Michel Bismut

Abstract

In the infinite dimensional space of Kähler potentials, the geodesic equation of disc type is a complex homogenous Monge–Ampère equation. The partial regularity theory established by Chen and Tian [C. R. Acad. Sci. Paris, Ser. I 340 (5) (2005)] amounts to an improvement of the regularity of the known $C^{1,1}$ solution to the geodesic of disc type to almost everywhere smooth. For such an almost smooth solution, we prove that the K-energy functional is sub-harmonic along such a solution. We use this to prove the uniqueness of extremal Kähler metrics and to establish a lower bound for the modified K-energy if the underlying Kähler class admits an extremal Kähler metric. ***To cite this article: X.X. Chen, G. Tian, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Unicité de métriques kählériennes extrémales. Dans l'espace de dimension infinie des potentiels de Kähler, l'équation géodésique de type disque est une équation de Monge–Ampère complexe homogène. Le résulat de régularité partielle établi dans cette note permet de renforcer le caractère $C^{1,1}$ de la solution obtenue antérieurement en montrant qu'elle est C^∞ presque partout. On démontre que la K-énergie est sous-harmonique sur une telle solution. On utilise ce résultat pour montrer l'unicité de la métrique de Kähler extrémale et pour établir une borne inférieure pour la K-énergie, quand la classe de Kähler sous-jacente admet une métrique Kählérienne extrémale. ***Pour citer cet article : X.X. Chen, G. Tian, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

In this Note and its sequel [5], we announce a new partial regularity theory for certain homogeneous complex Monge–Ampere equations and its application to extremal Kähler metrics in complex geometry. Details will appear

E-mail address: xxchen@math.wisc.edu (X. Chen).
[1] Both authors are supported by NSF research grants and the second author is also supported partially by a Simons fund grant.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.11.028

in [6]. Here we will discuss two main applications: uniqueness of extremal metrics and bounding the K-energy from below.

Following [3], we call a Kähler metric extremal if the complex gradient of its scalar curvature is a holomorphic vector field. In particular, any Kähler metric with constant scalar curvature is extremal; conversely, if the underlying Kähler manifold has no holomorphic vector fields, then an extremal Kähler metric is of constant scalar curvature. Our first result is

Theorem 1. *Let $(M, [\omega])$ be a compact Kähler manifold with a Kähler class $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$. Then there is at most one extremal Kähler metric with Kähler class $[\omega]$ modulo holomorphic transformations, that is, if ω_1 and ω_2 are extremal Kähler metrics with the same Kähler class, then there is a holomorphic transformation σ such that $\sigma^*\omega_1 = \omega_2$.*

The problem of uniqueness of extremal Kähler metrics has a long history. The uniqueness of Kähler–Einstein metrics was pointed out by Calabi in the early 1950s in the case of non-positive scalar curvature. In [1], Bando and Mabuchi proved the uniqueness of the Kähler–Einstein metric in the case of positive scalar curvature. In [16], Tian and Zhu proved the uniqueness of Kähler–Ricci Solitons. Following a suggestion of Donaldson, the first author proved in [4] the uniqueness of Kähler metrics with constant scalar curvature in any Kähler class which admits a Kähler metric with non-positive scalar curvature. In [8], Donaldson proved the uniqueness of constant scalar curvature Kähler metrics with rational Kähler class on any projective manifold (which is Kähler) without non-trivial holomorphic vector fields.[2]

As another consequence of our partial regularity theory, we will give a necessary condition on the existence of Kähler metrics with constant scalar curvature in terms of the K-energy. In [10], Mabuchi introduced the following functional $\mathbf{E}_\omega$ as follows: for any φ with $\omega_\varphi = \omega + \partial\bar{\partial}\varphi > 0$, define

$$\mathbf{E}_\omega(\varphi) = -\int_0^1 \int_M \dot{\varphi}\big(s(\omega_{\varphi_t}) - \mu\big)\omega_{\varphi_t}^n \wedge \mathrm{d}t,$$

where ω_{φ_t} is any path of Kähler metrics joining ω and ω_φ; $s(\omega_{\varphi_t})$ denotes the scalar curvature and μ is its average. It turns out that μ is determined by the first Chern class $c_1(M)$ and the Kähler class $[\omega]$.

Theorem 2. *Let M be a compact Kähler manifold with a constant scalar curvature Kähler metric ω. Then $\mathbf{E}_\omega(\varphi) \geqslant 0$ for any φ with $\omega_\varphi > 0$.*[3]

This theorem was proved for Kähler–Einstein metrics in [1] (also see [14]) and in [4] for Kähler manifolds with non-positive first Chern class. It can be also generalized to arbitrary extremal Kähler metrics by using the modified K-energy. It gives a partial answer to a conjecture of the second author: M has a constant scalar curvature Kähler metric in a given Kähler class $[\omega]$, if and only if, the K-energy is proper in a suitable sense on the space of Kähler metrics with the fixed Kähler class $[\omega]$. We will further discuss applications of the method used here to this problem on properness in a forthcoming paper. Combining Theorem 2 with results in [15] and [12], we can deduce:

[2] After we finished proving our uniqueness theorem, we learned that Mabuchi had posted a preprint [11] in which he extended Donaldson's arguments to any extremal Kähler metric with rational coefficients on any projective manifold.

[3] After the submission of this Note, we learned that an alternative proof of this theorem has been given by Donaldson in case of projective manifolds without nontrivial holomorphic vector fields [9].

Corollary 3. *Let (M, L) be a polarized algebraic manifold, that is, M is algebraic and L is a positive line bundle. If there is a constant scalar curvature Kähler metric with Kähler class equal to $c_1(L)$. Then (M, L) is asymptotically K-semistable or CM-semistable in the sense of* [14] (*also see* [15]).[4]

The proof of these two theorems is based on studying certain homogenous Complex Monge–Ampere equations (cf. (2)). These Monge–Ampere equations (cf. (2)) are related to the geodesic equation on the space of Kähler metrics with L^2-metric.

Let M be a compact Kähler manifold with a fixed Kähler metric ω and Σ be a Riemann surface with boundary $\partial\Sigma$. Let $\mathcal{H}_\omega$ denote the space of Kähler potentials

$$\mathcal{H}_\omega = \left\{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi = \omega + \partial\bar{\partial}\varphi > 0, \text{ on } M\right\}. \tag{1}$$

Suppose that ψ is a given function on $\partial\Sigma \times M$ such that $\psi(z, \cdot) \in \mathcal{H}_\omega (\forall z \in \Sigma)$. Consider the boundary value problem for function ϕ in $\Sigma \times M$ such that

$$\left(\pi_2^*\omega + \partial\bar{\partial}\phi\right)^{n+1} = 0 \quad \text{on } \Sigma \times M, \qquad \phi|_{\partial\Sigma\times M} = \psi, \tag{2}$$

where $\pi_2 : \Sigma \times M \to M$ is the projection and $\phi(z, \cdot) \in \mathcal{H}_\omega$ for any $z \in \Sigma$.

The proof of Theorems 1 and 2 starts with the following observations: given two functions φ_0 and φ_1 in $\mathcal{H}_\omega$, if there is a bounded smooth solution ϕ of (2) on $[0, 1] \times \mathbb{R} \times M$, then the evaluation function $f = \mathbf{E}(\phi(z, \cdot))$ is a bounded subharmonic function which is constant along each boundary component of $[0, 1] \times \mathbb{R}$; furthermore, if φ_0 is a critical metric of $\mathbf{E}$, then it follows from the Maximum principle that $\mathbf{E}(\varphi_1) \geqslant \mathbf{E}(\varphi_0)$ and equality holds if and only if each $\phi(z, \cdot)$ is a critical metric of $\mathbf{E}$. The infinite strip $[0, 1] \times \mathbb{R}$ can be approximated by discs $\Sigma_R = [0, 1] \times [-R, R]$ ($R \to \infty$). Hence, if we can show that Eq. (2) has a uniformly bounded solution for each Σ_R,[5] then Theorems 1 and 2 follow.

However, it is an extremely difficult problem to solve degenerate complex Monge–Ampere equations. Higher regularity was obtained by Cafferali, Kohn, Nirenberg and Spruck in 1980s for nondegenerate complex Monge–Ampere equations under certain convexity assumptions. Weak solutions for homogenous complex Monge–Ampere equations, say in L^p or $W^{1,p}$-norms, were extensively studied (cf. [2]). In [4], the first named author proved the following theorem, which plays a fundamental role in establishing our new results.

Theorem 4 [4]. *For any smooth map $\psi : \partial\Sigma \to \mathcal{H} = \mathcal{H}_\omega$,* (2) *always has a unique $C^{1,1}$-solution ϕ on $\Sigma \times M$ such that $\phi = \psi$ along $\partial\Sigma$.* [6] *Moreover, the $C^{1,1}$ bound of ϕ depends only on the C^2 bound of ψ.*

This result was used by the first author to prove Theorems 1 and 2 under an extra condition on the first Chern class $c_1(M)$. In order to remove this condition, we need to improve regularity of solutions in above theorem.

Historically, the solutions of homogeneous complex Monge–Ampere equations are closely related to foliations by holomorphic curves (cf. [7,13]; more references will be in [6]). In [13], Semmes formulated the Dirichlet problem for (2) in terms of a foliation by holomorphic curves with boundary in a totally real submanifold in the complex cotangent bundle of the underlying manifold. Establishing the existence of foliations by holomorphic disks with relatively mild singularity, we will show that for a generic boundary value ψ, there is an open subset in the moduli space of holomorphic discs with boundary which generates a foliation on $\Sigma \times M \backslash S$ where S is a closed subset of measure zero. To be more precise, we introduce the notion of almost smooth solutions.

[4] According to [12], the CM-stability (semistability) is equivalent to the K-stability (semistability).

[5] Σ_R has four corners, but we can easily smooth corners to obtain a smooth Riemann surface of disc type and use smoothed ones to approximate the given infinite strip.

[6] It is not known if $\phi(z, \cdot)$ lies in $\mathcal{H}$, but the first author proved that $\phi(z, \cdot)$ is always the limit of functions in $\mathcal{H}$.

Let ϕ be a solution of (2) constructed in Theorem 4. Define $\mathcal{R}_\phi \subset \Sigma \times M$ as the set of points where ω_ϕ is smooth Kähler form. Define $\mathcal{D}_\phi \subset T(\Sigma \times M)$ over $\mathcal{R}_\phi$ by

$$\mathcal{D}_\phi = \left\{ \frac{\partial}{\partial z} + v \in T_{(z,p)}(\Sigma \times M) \mid (z,p) \in \mathcal{R}_\phi, i_{\frac{\partial}{\partial z}+v}(\pi_2^*\omega + \partial\bar\partial\phi) = 0 \right\}. \tag{3}$$

Then $\mathcal{D}_\phi$ is a holomorphic integrable distribution. We say that a subset $\mathcal{R} \subset \mathcal{U} \subset \Sigma \times M$ is saturated in $\mathcal{U}$ if any maximal integral manifold of $\mathcal{D}$ in $\mathcal{R}$ is closed in $\mathcal{U}$.

Definition 5. A $C^{1,1}$ solution ϕ of (2) is almost smooth if it has the following properties:

(i) $\mathcal{R}_\phi$ is open, dense and saturated in $\Sigma \times M$ while the varying volume form $\omega^n_{\phi(z,\cdot)}$ extends to a continuous form on $\Sigma_0 \times M$, where $\Sigma_0 = (\Sigma \backslash \partial\Sigma)$. Moreover, this form is positive in $\mathcal{R}_\phi$ and vanishes identically on its complement.
(ii) The distribution $\mathcal{D}_\phi$(cf. Eq. (3)) extends to a continuous distribution over a saturated set $\tilde{\mathcal{V}}_\phi$ of $\Sigma_0 \times M$ such that its complement $\tilde{\mathcal{S}}_\phi$ is locally extendable.[7] The set $\tilde{\mathcal{S}}_\phi$ is referred as the singular set of ϕ.
(iii) The leaf vector field $\frac{\partial}{\partial z} + v$ of $\mathcal{D}_\phi$ is uniformly bounded in $\tilde{\mathcal{V}}_\phi$.

Theorem 6. *Let Σ be a disk in $\mathbb{C}$. For a generic boundary map $\psi : \partial\Sigma \to \mathcal{H}_\omega$, there exists a unique almost smooth solution of* (2).

By using the properties of the K-energy, one can prove Theorems 1 and 2 by using such almost smooth solutions and following the arguments described above.

References

[1] S. Bando, T. Mabuchi, Uniqueness of Kähler Einstein metrics modulo connected group actions, in: Algebraic Geometry, in: Adv. Stud. Pure Math., 1987, pp. 11–40.
[2] E.D. Bedford, T.A. Taylor, The Drichelet problem for the complex Monge–Ampere operator, Invent. Math. 37 (1976) 1–44.
[3] E. Calabi, Extremal Kähler metrics, in: Seminar on Differential Geometry, in: Ann. of Math. Stud., vol. 102, Princeton University Press, 1982, pp. 259–290.
[4] X.X. Chen, Space of Kähler metrics, J. Differential Geom. 56 (2000) 189–234.
[5] X.X. Chen, G. Tian, Partial regularity of homogeneous complex Monge–Ampere equations, C. R. Acad. Sci. Paris, Ser. I 340 (5) (2005).
[6] X.X. Chen, G. Tian, Geometry of Kähler metrics and foliations by holomorphic discs, Preprint, 2004.
[7] S.K. Donaldson, Holomorphic discs and the complex Monge–Ampere equation, J. Symplectic Geometry 1 (2002) 171–196.
[8] S.K. Donaldson, Scalar curvature and projective embeddings. I, J. Differential Geom. 59 (2001) 479–522.
[9] S.K. Donaldson, Scalar curvature and projective embeddings. II, math.DG/0407534.
[10] T. Mabuchi, Some Symplectic geometry on compact Kähler manifolds I, Osaka J. Math. 24 (1987) 227–252.
[11] T. Mabuchi, Stability of extremal Kähler manifolds, math.DG/0404211.
[12] S. Paul, G. Tian, Algebraic and analytic K-stability, Preprint, 2004.
[13] S. Semmes, Complex Monge–Ampere and sympletic manifolds, Amer. J. Math. 114 (1992) 495–550.
[14] G. Tian, Kähler–Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997) 1–39.
[15] G. Tian, Bott–Chern forms and geometric stability, Discrete Contin. Dynam. Systems 6 (2000) 1–39.
[16] G. Tian, X.H. Zhu, Uniqueness of Kähler–Ricci solitons, Acta Math. 184 (2000) 271–305.

[7] A closed subset $S \subset \Sigma \times M$ is *locally extendable* if for any continuous function in $\Sigma \times M$ which is $C^{1,1}$ on $\Sigma \times M \setminus S$ can be extended to a $C^{1,1}$ function on $\Sigma \times M$. Note that any subset of codimension 2 or higher is automatically locally extendable.