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Numerical Analysis

## Lower bounds for additive Schwarz methods with mortars

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### Abstract

We establish lower bounds for the condition number of overlapping additive Schwarz algorithms for elliptic problems discretized by mortar finite elements. These bounds coincide, up to constants, with the classical upper bounds from the literature. The optimality of the condition number estimates is thus established. *To cite this article: D. Stefanica, C. R. Acad. Sci. Paris, Ser. I 339 (2004).*

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### Résumé

**Bornes inférieures pour méthode de Schwarz additives et éléments finis avec joints.** On détermine des bornes inférieures de conditionnement d'algorithmes de type Schwarz additif pour des problèmes elliptiques discrétisés par éléments finis avec joints. Ces limites sont identiques à des constantes multiplicatives près, aux bornes supérieures classiques établies par ailleurs. L'optimalité du conditionnement est ainsi démontré. *Pour citer cet article : D. Stefanica, C. R. Acad. Sci. Paris, Ser. I 339 (2004).*

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### Version française abrégée<sup>1</sup>

Nous montrons que les bornes supérieures [5] du conditionnement d'algorithmes additifs de Schwarz pour l'équation de Poisson avec condition aux limites de Neumann–Dirichlet sur un domaine  $\Omega$  sont optimales. Ce problème est discrétisé en utilisant un espace  $V$  d'éléments finis du premier ordre avec joints sur une décomposition polygonale  $\{D_k\}_{k=1:K}$  de  $\Omega$ .

Pour construire un opérateur de Schwarz pour le problème discret (1), nous utilisons une décomposition de recouvrement  $\{\Omega_i\}_{i=1:N}$  de  $\Omega$ , avec chevauchement  $\delta$ , et un espace  $V_0$  fait d'éléments  $P^1$  ou  $Q^1$ , de diamètre  $H$ .

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<sup>1</sup> Les références font référence à la version anglaise.

La restriction de  $V$  à  $\Omega_i$  est notée  $V_i$  et la restriction aux  $V_i \times V_i$  de  $a^\Gamma$ , le produit  $H^1$  décomposé sur  $\{D_k\}_{k=1:K}$ , est notée  $a_i^\Gamma$ ,  $i = 1 : N$ . L'opérateur de Schwarz additif  $T_{as}$  est reconstitué à partir des opérateurs des solutions locales  $T_i$  correspondant à  $a_i^\Gamma$  comme suit :  $T_{as} = T_0 + T_1 + \dots + T_N$  ; voir aussi (2).

Pour un petit chevauchement ( $\delta/H < 1/4$ ), il est bien connu que  $\kappa(T_{as}) \leq C(1 + H/\delta)$  ; voir [4] pour les éléments finis continus et [5] pour les éléments avec joints. Nous montrons que  $\kappa(T_{as}) \geq C(1 + H/\delta)$  pour les éléments avec joints.

De (8), il suit immédiatement que  $\lambda_{\max}(T_{as}) \geq 1$ . Pour montrer que  $\lambda_{\min}(T_{as}) \leq C\delta/H$ , voir (9), nous construisons une fonction  $v \in V$  telle que pour toute décomposition  $v = v_0 + v_1 + \dots + v_N$ ,  $v_i \in V$ , l'inégalité  $a^\Gamma(v, v) \leq C \frac{\delta}{H} (a_1^\Gamma(v_1, v_1) + a_2^\Gamma(v_2, v_2))$  est satisfaite.

## 1. Introduction

Mortar finite elements, first introduced by Bernardi, Maday and Patera in [1], are nonconforming finite elements that allow for a geometrically nonconforming decomposition of the computational domain and, at the same time, for the optimal coupling of different variational approximations in different subregions. Mortar elements are well suited for parallel computing and have several advantages over the conforming finite elements, e.g., flexible mesh generation and straightforward local refinement. Many domain decomposition algorithms have been extended to mortar element discretization in order to take advantage of the inherent flexibility of the mortars, without any loss in convergence properties.

We establish here that the upper bounds [5] on the condition numbers of the overlapping additive Schwarz algorithms for mortars are sharp. To do so, we prove estimates for the lower bounds on these condition numbers using methods pioneered by Brenner [2]. We restrict our attention to the two dimensional Poisson equation with mixed Neumann–Dirichlet boundary conditions, discretized by first order mortar elements on a polygonal domain  $\Omega$ . The extensions to three dimensional curvilinear domains, higher order mortar elements, and self-adjoint elliptic equations are standard.

Let  $f \in L^2(\Omega)$ . We look for a solution  $u \in H^1(\Omega)$  to the problem  $-\Delta u = f$  on  $\Omega$ , with  $u = 0$  on  $\partial\Omega_D$  and  $\partial u/\partial n = 0$  on  $\partial\Omega_N$ , where  $\partial\Omega = \partial\Omega_N \cup \partial\Omega_D$  and  $\partial\Omega_D$  has positive Lebesgue measure. This problem is discretized using a mortar finite element space  $V$  on a nonoverlapping polygonal partition  $\{D_k\}_{k=1:K}$  of  $\Omega$ ; see Section 2 for details on the definition of  $V$ . The resulting discrete problem consists of finding  $u_h \in V$  such that

$$a^\Gamma(u_h, v_h) = f(v_h), \quad \forall v_h \in V, \quad (1)$$

where  $a^\Gamma(w_h, v_h) = \sum_k \int_{D_k} \nabla w_h \cdot \nabla v_h \, dx$  and  $f(v_h) = \int_\Omega f v_h \, dx$ . The existence and uniqueness of the solution  $u_h$  of (1) follows from the Lax–Milgram lemma and a Friedrichs inequality for mortar finite elements.

To assemble the additive Schwarz preconditioner, we introduce an overlapping decomposition  $\{\Omega_i\}_{i=1:N}$  of  $\Omega$ . Let  $\delta$  be the maximum overlap between the subdomains  $\{\Omega_i\}_{i=1:N}$ . Let  $V_i \subset V$ ,  $i = 1 : N$ , be local spaces made of the mortar functions  $v_i \in V$  which vanish at all the genuine degrees of freedom outside  $\Omega_i$ . A coarse space  $V_0$  made of  $P^1$  or  $Q^1$  elements of diameter  $H$  is introduced on  $\Omega$ . It may be unstructured with respect to both the mortar partition and the overlapping partition. Without a coarse space, the number of iterations to convergence would depend on the number of overlapping subdomains.

Let  $I_i : V_i \rightarrow V$  be embedding operators,  $i = 0 : N$ , let  $a_0^\Gamma : V_0 \times V_0 \rightarrow R$  be the  $H^1$  inner-product, and let  $a_i^\Gamma : V_i \times V_i \rightarrow R$  be the restriction of  $a^\Gamma$  to  $V_i \times V_i$ ,  $i = 1 : N$ . The corresponding projection-like operators  $\tilde{T}_i : V \rightarrow V_i$  are defined by:

$$a_i^\Gamma(\tilde{T}_i v, v_i) = a^\Gamma(v, I_i v_i), \quad \forall v_i \in V_i, v \in V, i = 0 : N. \quad (2)$$

Let  $T_i = I_i \tilde{T}_i$  and  $T_{as} = T_0 + T_1 + \dots + T_N$ . The additive Schwarz method for (1) consists of solving  $T_{as} u = g_{as}$  without further preconditioning by a conjugate gradient algorithm; here  $g_{as}$  can be computed by solving local problems without previous knowledge of  $u$ .

For small overlap, i.e.,  $\delta/H < 1/4$ , the following upper bound for the condition number  $\kappa(T_{as})$  of the additive Schwarz operator was proved in [4] for conforming finite elements, and for mortar finite elements in [5]:

$$\kappa(T_{as}) \leq C \left( 1 + \frac{H}{\delta} \right). \tag{3}$$

Here, and throughout the rest of the paper,  $C$  denotes a constant independent of any parameters of the problem, e.g.,  $h, H, \delta, K, N$ .

**2. Special choice of mortar functions**

The bound (3) was shown in [2] to be sharp for conforming finite elements. To prove a similar result for mortars, we introduce a special partition of the computational domain with a structured coarse space. Our construction can be extended to the nonconforming case with unstructured overlapping subdomains and coarse space.

Let  $\{D_k\}_{k=1:K}$  be a geometrically conforming mortar decomposition of a polygonal domain  $\Omega$  into rectangles of diameter of order  $H$ . Let  $\{\Omega_k\}_{k=1:K}$  be the overlapping subdomains, where  $\Omega_k$  is the smallest rectangle obtained by including all the nodes at distance  $\delta/2$  from  $\partial D_k$ . The diameter of the overlapping subdomains is thus of order  $H$ . The coarse space  $V_0$  is the  $Q^1$  finite element space defined on the coarse grid determined by the corner nodes of  $\{D_k\}_{k=1:K}$ .

The restriction of the mortar finite element space  $V$  to any rectangle  $D_k$  is a  $Q_1$  finite element function on a mesh of diameter  $h$ . Weak continuity is required across  $\Gamma$ , the interface between the subregions  $\{D_k\}_{k=1:K}$ . We choose a set of edges of  $\{D_k\}_{k=1:K}$ , called nonmortars, which form a disjoint partition of  $\Gamma$ . For each nonmortar side  $\gamma$  there exists exactly one side opposite to it, which is called a mortar side. The jump of a mortar function  $w \in V$  across any nonmortar  $\gamma$  must be orthogonal to a space of test functions  $\Psi(\gamma)$ :

$$\int_{\gamma} [w]\psi \, ds = 0, \quad \forall \psi \in \Psi(\gamma). \tag{4}$$

In [1],  $\Psi(\gamma)$  was chosen to be a subspace of codimension two of  $V(\gamma)$ , the restriction of  $V$  to  $\gamma$ . It consists of continuous, piecewise linear functions on  $\gamma$  that are constant in the first and last mesh intervals of  $\gamma$ . A biorthogonal mortar space was introduced in [3], corresponding to discontinuous piecewise linear test functions.

For either mortar space, the values of  $w$  at the end points of  $\gamma$  cannot be determined from the mortar conditions (4) and therefore are genuine degrees of freedom. We denote by  $\pi_{\alpha,\beta} : L^2(\gamma) \rightarrow V(\gamma)$  the mortar projection operator corresponding to (4), which takes values  $\alpha$  and  $\beta$ , respectively, at the end points  $A$  and  $B$  of  $\gamma$ .

It is easy to see that

$$\pi_{\alpha_1,\beta_1}(w_1) - \pi_{\alpha_2,\beta_2}(w_2) = \pi_{\alpha_1-\alpha_2,\beta_1-\beta_2}(w_1 - w_2), \tag{5}$$

for all  $w_i \in V$ ,  $\alpha_i, \beta_i \in R$ ,  $i = 1, 2$ . Note that  $\pi_{0,0}$ , the mortar projection operator which vanishes at the end points of  $\gamma$ , is stable in  $L^2(\gamma)$ ,  $H_0^1(\gamma)$ , and, by interpolation, in  $H_{00}^{1/2}(\gamma)$ ; see, e.g., [3].

To establish an upper bound on  $\lambda_{\min}(T_{as})$ , we introduce a special mortar function  $v \in V$ . Following a construction from [2], let  $\hat{v} \in H^1([0, 1])$  be a piecewise linear function on the uniform mesh of size  $1/8$  on  $[0, 1]$  such that  $\hat{v}$  vanishes outside  $(1/4, 3/4)$  and is orthogonal in the  $L^2([0, 1])$  inner product to any linear function. We choose  $v \in V$  to be discrete harmonic on every subregion  $D_k$  and vanish on  $\Gamma$  except on  $\zeta$  and  $\gamma$ , opposite mortar and nonmortar sides of subregions  $D_1$  and  $D_2$ , respectively. We also choose the local mesh on  $D_1$  such that its restriction to  $\zeta$  is a dyadic subdivision of  $\zeta$ . Let  $A$  and  $B$  be the common end points of  $\zeta$  and  $\gamma$ , and let  $Q_1$  and  $Q_2$  be the mesh nodes on  $AB$  obtained by scaling from the nodes  $1/4$  and  $3/4$  on  $[0, 1]$ . The restriction  $v_{\zeta}$  of  $v$  to  $\zeta$  is

obtained by scaling  $\hat{v}$ , while  $v_\gamma$ , the restriction of  $v$  to  $\gamma$ , is obtained via a mortar projection with 0 values at  $A$  and  $B$ , i.e.,  $v_\gamma = \pi_{0,0}(v_\zeta)$ . Note that  $v_\zeta$  is defined by scaling  $\hat{v}$  and has compact support  $Q_1 Q_2 \subset [0, 1]$ . Therefore,

$$\|v_\zeta\|_{H_{00}^{1/2}(\zeta)}^2 = |v_\zeta|_{H^{1/2}(\zeta)}^2 = \frac{C}{H} \|v_\zeta\|_{L^2(\zeta)}^2 = \frac{C}{H} \|v_\zeta\|_{L^2(Q_1 Q_2)}^2. \tag{6}$$

Since  $v$  is discrete harmonic and using the stability properties of  $\pi_{0,0}$ , we find that

$$a^\Gamma(v, v) = \|v_\zeta\|_{H_{00}^{1/2}(\zeta)}^2 + \|v_\gamma\|_{H_{00}^{1/2}(\gamma)}^2 \leq C \|v_\zeta\|_{H_{00}^{1/2}(\zeta)}^2 = \frac{C}{H} \|v_\zeta\|_{L^2(Q_1 Q_2)}^2. \tag{7}$$

### 3. Lower bound on the condition number

We recall that

$$\lambda_{\max}(T_{as}) = \max_{v \in V, v \neq 0} \frac{a^\Gamma(v, v)}{\min_{v=I_0 v_0 + \dots + I_N v_N} \sum_{i=0}^N a_i^\Gamma(v_i, v_i)} \tag{8}$$

$$\lambda_{\min}(T_{as}) = \min_{v \in V, v \neq 0} \frac{a^\Gamma(v, v)}{\min_{v=I_0 v_0 + \dots + I_N v_N} \sum_{i=0}^N a_i^\Gamma(v_i, v_i)}. \tag{9}$$

The lower bound  $\lambda_{\max}(T_{as}) \geq 1$  follows immediately by choosing  $v$  in the space  $V_1$  and a decomposition of  $v$  with  $v_1 = v$  and  $v_i = 0$  for  $i \neq 1$ .

To prove an upper bound of the form  $\lambda_{\min}(T_{as}) \leq C\delta/H$ , it is enough to show that, for any decomposition of the special function  $v \in V$  from Section 2 of the form  $v = v_0 + v_1 + \dots + v_N$ , with  $v_i \in V_i, i = 0 : N$ , the following inequality holds:  $a^\Gamma(v, v) \leq C \frac{\delta}{H} (a_1^\Gamma(v_1, v_1) + a_2^\Gamma(v_2, v_2))$ . Using (7), we only need to show that

$$\|v_\zeta\|_{L^2(Q_1 Q_2)}^2 \leq C\delta (a_1^\Gamma(v_1, v_1) + a_2^\Gamma(v_2, v_2)). \tag{10}$$

Let  $v_{i,\zeta}$  and  $v_{i,\gamma}$  be the restrictions of  $v_i$  to  $\zeta$  and  $\gamma$ , respectively, for  $i = 1, 2$ . Note that  $Q_1 Q_2$  belongs to only two overlapping subdomains,  $\Omega_1$  and  $\Omega_2$ , since  $\delta/H \leq 1/4$ . Therefore, on  $Q_1 Q_2$ ,  $v_\zeta = v_0 + v_1 + v_2$ . Recall that  $v_0$  is linear on  $\zeta$ , and that, by construction,  $v_\zeta$  is  $L^2$ -orthogonal on  $\zeta$  to any linear function. Using the Schwarz inequality, we obtain:

$$\|v_\zeta\|_{L^2(Q_1 Q_2)}^2 \leq 2(\|v_{1,\zeta}\|_{L^2(Q_1 Q_2)}^2 + \|v_{2,\zeta}\|_{L^2(Q_1 Q_2)}^2). \tag{11}$$

By definition,  $v_2 \in V_2$  vanishes on  $\partial\Omega_2$ , i.e., at distance  $\delta/2$  from  $Q_1 Q_2$ . From the Friedrichs inequality on  $D_1 \cap \Omega_2$ , we find that

$$\|v_{2,\zeta}\|_{L^2(Q_1 Q_2)}^2 \leq C\delta a_2^\Gamma(v_2, v_2). \tag{12}$$

Let  $I_\gamma : L^2(AB) \rightarrow V(\gamma)$  be the piecewise linear interpolation onto  $\gamma$ . Then

$$v_{1,\zeta} = (v_{1,\zeta} - I_\gamma(v_{1,\zeta})) + (I_\gamma(v_{1,\zeta}) - v_{1,\gamma}) + v_{1,\gamma}. \tag{13}$$

From a classical interpolation inequality, we find that

$$\|v_{1,\zeta} - I_\gamma(v_{1,\zeta})\|_{L^2(Q_1 Q_2)}^2 \leq Ch |v_1|_{H^1(D_1)}^2. \tag{14}$$

Since  $v_1 \in V_1$  vanishes at distance  $\delta/2$  from  $Q_1 Q_2$ , we use Friedrichs inequality to obtain:

$$\|v_{1,\gamma}\|_{L^2(Q_1 Q_2)}^2 \leq C\delta a_1^\Gamma(v_1, v_1). \tag{15}$$

To estimate the  $L^2(Q_1 Q_2)$  norm of  $I_\gamma(v_{1,\zeta}) - v_{1,\gamma}$ , recall that  $v_{1,\gamma} = \pi_{v_{1,\gamma}(A), v_{1,\gamma}(B)}(v_{1,\zeta})$ . Since  $I_\gamma(v_{1,\zeta})$  is piecewise linear on  $\gamma$ , its mortar projection on  $\gamma$  with values  $v_{1,\zeta}(A)$  and  $v_{1,\zeta}(B)$  at  $A$  and  $B$  is itself, i.e.,

$I_\gamma(v_{1,\zeta}) = \pi_{v_{1,\zeta}(A), v_{1,\zeta}(B)}(I_\gamma(v_{1,\zeta}))$ . Let  $[v_1] = v_{1,\zeta} - v_{1,\gamma}$  be the jump of  $v_1$  across  $\gamma$ . Applying (5) repeatedly, we find that

$$I_\gamma(v_{1,\zeta}) - v_{1,\gamma} = \pi_{[v_1](A), [v_1](B)}(I_\gamma(v_{1,\zeta}) - v_{1,\gamma}) = \pi_{0,0}(I_\gamma(v_{1,\zeta}) - v_{1,\gamma}) + \pi_{[v_1](A), [v_1](B)}(0). \quad (16)$$

From the stability of the mortar projection  $\pi_{0,0}$  and an interpolation estimate, we find that

$$\|\pi_{0,0}(I_\gamma(v_{1,\zeta}) - v_{1,\gamma})\|_{L^2(Q_1 Q_2)}^2 \leq C \|v_{1,\zeta} - I_\gamma(v_{1,\zeta})\|_{L^2(AB)}^2 \leq Ch |v_1|_{H^1(D_1)}^2. \quad (17)$$

For biorthogonal mortars, the mortar projection is a local operator, and thus  $\pi_{[v_1](A), [v_1](B)}(0)$  vanishes on  $Q_1 Q_2$ . For classical mortars, the nodal values of  $\pi_{[v_1](A), 0}(0)$  and  $\pi_{0, [v_1](B)}(0)$  on  $Q_1 Q_2$  decline exponentially. From (5), it is easy to see that  $\pi_{[v_1](A), [v_1](B)}(0) = \pi_{[v_1](A), 0}(0) + \pi_{0, [v_1](B)}(0)$ . We use Friedrichs inequality and a Sobolev-type inequality for finite element functions to bound  $[v_1](A)$  and  $[v_1](B)$  in terms of the  $H^1$  seminorm of  $v_1$  on  $D_1$  and  $D_2$ . We obtain:

$$\|\pi_{[v_1](A), [v_1](B)}(0)\|_{L^2(Q_1 Q_2)}^2 \leq Ch (|v_1|_{H^1(D_1)}^2 + |v_1|_{H^1(D_2)}^2) \leq Cha_1^\Gamma(v_1, v_1). \quad (18)$$

We conclude from (16)–(18) that  $\|I_\gamma(v_{1,\zeta}) - v_{1,\gamma}\|_{L^2(Q_1 Q_2)}^2 \leq Cha_1^\Gamma(v_1, v_1)$ , and, using (13)–(15), that

$$\|v_{1,\zeta}\|_{L^2(Q_1 Q_2)}^2 \leq C\delta a_1^\Gamma(v_1, v_1). \quad (19)$$

From (11), (12) and (19), we find that  $\|v_\zeta\|_{L^2(Q_1 Q_2)}^2 \leq C\delta(a_1^\Gamma(v_1, v_1) + a_2^\Gamma(v_2, v_2))$ . Thus (10) holds true and therefore, as explained above,  $\lambda_{\min}(T_{as}) \leq C\delta/H$ . Since  $1 \leq \lambda_{\max}(T_{as})$ , we conclude that  $C(1 + H/\delta) \leq \kappa(T_{as})$ .

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