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Algebra

On the intersection of null spaces for matrix substitutions in a non-commutative rational formal power series

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Abstract

For a rational formal power series in N non-commuting indeterminates, with matrix coefficients, we establish the formula which relates the intersection of the null spaces of coefficients to the intersection of the null spaces of values of this series at N -tuples of $n \times n$ matrices, for n large enough. As an application, we formulate the criteria of observability, controllability, and minimality for a recognizable formal power series representation in terms of matrix substitutions. **To cite this article:** *D. Alpay, D.S. Kalyuzhnyi-Verbovetzkiĭ, C. R. Acad. Sci. Paris, Ser. I 339 (2004).*

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Résumé

Sur l'intersection des noyaux des opérateurs de substitution de matrices pour les séries formelles non commutatives. Nous démontrons une formule qui relie l'intersection des noyaux des coefficients d'une série formelle de N variables non commutatives à l'intersection des noyaux des valeurs de cette série pour des N -uplets de matrices d'ordre n pour n assez grand. Comme application, nous donnons des critères d'observabilité, de commandabilité et de minimalité en termes de représentations pour les séries formelles reconnaissables et en utilisant des substitutions matricielles. **Pour citer cet article :** *D. Alpay, D.S. Kalyuzhnyi-Verbovetzkiĭ, C. R. Acad. Sci. Paris, Ser. I 339 (2004).*

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La théorie des séries formelles non commutatives apparaît dans différents domaines, par exemple en algèbre et en combinatoire. Les séries rationnelles (ou reconnaissables ; par le théorème de Kleene–Schützenberger [8,12] ces

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deux notions sont équivalentes) ont été utilisées abondamment en théorie des automates et des langages formels ; voir [4]. En théorie des systèmes, la réalisation des séries rationnelles non commutatives fut étudiée par Fliess (voir [5,6]). Cette dernière décennie, il a été observé que les propriétés des séries formelles rationnelles et des polynômes non commutatifs sont liées de manière très forte aux opérations associées d'évaluation (matricielle ou à valeurs opérateurs). La substitution par des opérateurs de ℓ_2 permet l'étude de la stabilité de systèmes dont les incertitudes ont une certaine structure ; voir [2,3]. Les substitutions par des matrices d'ordre quelconque (ou d'ordre suffisamment grand) ont été utilisées pour l'étude d'inégalités matricielles : en particulier, elles permettent de les réduire à des inégalités matricielles linéaires (*linear matrix inequalities, LMI's*), à des représentations de polynômes matriciels positifs en termes de sommes de carrés de polynômes (*sums of squares, SoS*), à des problèmes de convexité dans le cadre non commutatif, etc. Nous renvoyons à l'article de revue de Helton [7] pour plus de détails et des références. Notre Note est de même nature que les travaux cités plus haut. La formule que nous démontrons pour les noyaux des substitutions matricielles (Théorème 3.1) donne un nouvel outil pour l'étude des séries formelles rationnelles non commutatives. Le Théorème 3.3 donne une application en théorie des systèmes. Nous remarquons que ces résultats deviennent beaucoup plus simples pour le cas d'une seule variable ; voir [1].

1. Introduction

Let \mathbb{F}_N denote the free semigroup with N generators g_1, \dots, g_N (the alphabet), and the neutral element \emptyset (the empty word). The multiplication law in \mathbb{F}_N is concatenation: the product of any two of its elements (words in the alphabet g_1, \dots, g_N), $w = g_{j_1} \cdots g_{j_m}$ and $w' = g_{k_1} \cdots g_{k_{m'}}$, is $ww' = g_{j_1} \cdots g_{j_m} g_{k_1} \cdots g_{k_{m'}}$, and $w\emptyset = \emptyset w = w$. The length of a word $w = g_{j_1} \cdots g_{j_m}$ is $|w| = m$, and $|\emptyset| = 0$. The transpose of a word $w = g_{j_1} \cdots g_{j_m}$ is $w^T = g_{j_m} \cdots g_{j_1}$, and $\emptyset^T = \emptyset$. For such a word w and indeterminates $z = (z_1, \dots, z_N)$, set $z^w = z_{j_1} \cdots z_{j_m}$, and $z^\emptyset = 1$. Let $\mathbb{C}^{p \times q}$ stand for the linear space of complex $p \times q$ matrices, and $\mathbb{C}^{p \times q} \langle z_1, \dots, z_N \rangle$ stand for the linear space of formal polynomials

$$P(z) = \sum_{w \in \mathbb{F}_N: |w| \leq m} P_w z^w,$$

with some $m \in \mathbb{Z}_+$ and coefficients $P_w \in \mathbb{C}^{p \times q}$. In the case $p = q = 1$, the space $\mathbb{C} \langle z_1, \dots, z_N \rangle$ is a non-commutative ring. Similarly, the linear space $\mathbb{C}^{p \times q} \langle \langle z_1, \dots, z_N \rangle \rangle$ of formal power series

$$F(z) = \sum_{w \in \mathbb{F}_N} F_w z^w,$$

with coefficients $F_w \in \mathbb{C}^{p \times q}$, in the case $p = q = 1$ turns into a non-commutative ring, $\mathbb{C} \langle \langle z_1, \dots, z_N \rangle \rangle$. A formal power series $F(z) \in \mathbb{C} \langle \langle z_1, \dots, z_N \rangle \rangle$ is invertible if and only if $F_\emptyset \neq 0$.

Let $\mathbb{C} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$ be the minimal subring in $\mathbb{C} \langle \langle z_1, \dots, z_N \rangle \rangle$ which contains all formal polynomials and inverses of polynomials (when they exist). Elements of this subring are called rational formal power series with coefficients in \mathbb{C} . The subspace $\mathbb{C}^{p \times q} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$ in $\mathbb{C}^{p \times q} \langle \langle z_1, \dots, z_N \rangle \rangle$ consists of formal power series (which are called rational) $F(z)$ such that their matrix entries $F(z)_{i,j}$, $i = 1, \dots, p$, $j = 1, \dots, q$, belong to $\mathbb{C} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$. The Kleene–Schützenberger theorem [8,12] (see also Fliess [5]) says that a formal power series $F(z) \in \mathbb{C}^{p \times q} \langle \langle z_1, \dots, z_N \rangle \rangle$ is rational if and only if $F(z)$ is recognizable, i.e., there exist $r \in \mathbb{N}$, $A = (A_1, \dots, A_N) \in (\mathbb{C}^{r \times r})^N$, $B \in \mathbb{C}^{r \times q}$, and $C \in \mathbb{C}^{p \times r}$ such that $F_w = CA^w B$, $\forall w \in \mathbb{F}_N$ (here for $w = g_{j_1} \cdots g_{j_m} \in \mathbb{F}_N$ one sets $A^w = A_{j_1} \cdots A_{j_m}$ and $A^\emptyset = I_r$), or equivalently,

$$F(z) = \sum_{w \in \mathbb{F}_N} CA^w B z^w = C \sum_{m=0}^{\infty} \left(\sum_{k=1}^N A_k z_k \right)^m B = C \left(I_r - \sum_{k=1}^N A_k z_k \right)^{-1} B. \quad (1)$$

Fliess has shown in [5] that the minimal possible $r \in \mathbb{N}$ for such a representation is equal to the rank of the (infinite) Hankel matrix $\mathcal{H}_F = (F_{ww^T})_{w, w' \in \mathbb{F}_N}$ where the words in \mathbb{F}_N are ordered lexicographically.

For any $n \in \mathbb{N}$ and $Z = (Z_1, \dots, Z_N) \in (\mathbb{C}^{n \times n})^N$ one can consider the substitution in a rational formal power series $F(z) = \sum_{w \in \mathbb{F}_N} F_w z^w$:

$$F(Z) = \sum_{w \in \mathbb{F}_N} F_w \otimes Z^w.$$

Due to representation (1), this series converges in some neighborhood of $Z = 0$. Moreover, if $\|X\|$ is the standard norm (i.e., the operator (2, 2)-norm) of an $X \in \mathbb{C}^{n \times n}$, such a neighborhood can be chosen as

$$\Gamma_n(\varepsilon) = \{Z \in (\mathbb{C}^{n \times n})^N : \|Z_k\| < \varepsilon, k = 1, \dots, N\},$$

where ε is independent of n . For a fixed n , one can consider $F(Z)$ as a rational matrix-valued function of scalar variables $(Z_k)_{i,j}$, $k = 1, \dots, N$, $i, j = 1, \dots, n$, which is holomorphic on $\Gamma_n(\varepsilon)$. Thus, any representation of $F(z)$ by finite combinations of sums and products of polynomials and inverses of polynomials for all entries $F(z)_{i,j}$, $i = 1, \dots, p$, $j = 1, \dots, q$, defines $F(Z)$ on some open and dense subset of $(\mathbb{C}^{n \times n})^N$.

In this Note we establish the relation between the spaces $\bigcap_{w \in \mathbb{F}_N} \ker F_w$ and $\bigcap_{Z \in \Gamma_n(\varepsilon)} \ker F(Z)$ for a rational formal power series $F(z)$. We will start with the special case of formal polynomials (i.e., finite formal power series), and then extend our results to the general case.

2. The case of formal polynomials

Theorem 2.1. *Let $P(z) = \sum_{|w| \leq m} P_w z^w \in \mathbb{C}^{p \times q}(z_1, \dots, z_N)$. Then for every $n \in \mathbb{N}$: $n \geq m^m$ (in the case $m = 0$, for every $n \in \mathbb{N}$),*

$$\bigcap_{Z \in (\mathbb{C}^{n \times n})^N} \ker P(Z) = \left(\bigcap_{w \in \mathbb{F}_N: |w| \leq m} \ker P_w \right) \otimes \mathbb{C}^n, \tag{2}$$

and moreover, there exist an $l \in \mathbb{N}$: $l \leq qn$, and N -tuples of matrices $Z^{(1)}, \dots, Z^{(l)} \in (\mathbb{C}^{n \times n})^N$ such that

$$\bigcap_{j=1}^l \ker P(Z^{(j)}) = \left(\bigcap_{w \in \mathbb{F}_N: |w| \leq m} \ker P_w \right) \otimes \mathbb{C}^n.$$

Proof. For $m = 0$, equality (2) is obvious for any $n \in \mathbb{N}$. For $m > 0$, the inclusion “ \supset ” in (2) is also obvious for any $n \in \mathbb{N}$. Let us show that for $m > 0$ and $n = m^m$, the inclusion “ \subset ” in (2) is valid.

For $m = n = 1$, one has $P(z) = P_\emptyset + \sum_{k=1}^N P_{g_k} z_k$. If $h \in \bigcap_{Z \in \mathbb{C}^N} \ker P(Z)$ then, clearly, $P_\emptyset h = P_{g_1} h = \dots = P_{g_N} h = 0$, which means the inclusion “ \subset ” in (2) for this case.

For $m > 1$ and $n = m^m$, let the matrix $S \in \mathbb{C}^{n \times n} \cong (\mathbb{C}^{m \times m})^{\otimes m}$ be defined by its action on basis vectors $e_{i_1} \otimes \dots \otimes e_{i_m} \in (\mathbb{C}^m)^{\otimes m}$ (here e_j , $j = 1, \dots, m$, is the standard basis of \mathbb{C}^m) as follows:

$$S(e_{i_1} \otimes \dots \otimes e_{i_m}) = e_{i_m} \otimes e_{i_1} \otimes \dots \otimes e_{i_{m-1}}.$$

Thus, S is a permutation matrix. Define $Z' = (Z'_1, \dots, Z'_N) \in (\mathbb{C}^{n \times n})^N \cong ((\mathbb{C}^{m \times m})^{\otimes m})^N$ by

$$Z'_j = (M_j \otimes I_m^{\otimes m-1})S, \quad j = 1, \dots, N,$$

where $M_j \in \mathbb{C}^{m \times m}$, $j = 1, \dots, N$, are arbitrary. Let $h \in \bigcap_{Z \in (\mathbb{C}^{n \times n})^N} \ker P(Z)$. Then

$$\begin{aligned} 0 &= P(Z')h = \left(\sum_{w \in \mathbb{F}_N: |w| \leq m} P_w \otimes Z'^w \right) \left(\sum_{1 \leq i_1, \dots, i_m \leq m} h_{i_1, \dots, i_m} \otimes e_{i_1} \otimes \dots \otimes e_{i_m} \right) \\ &= \sum_{w=g_{j_1} \dots g_{j_{|w|}}: |w| \leq m} \sum_{1 \leq i_1, \dots, i_m \leq m} P_w h_{i_1, \dots, i_m} \otimes M_{j_1} e_{i_{m-|w|+1}} \otimes \dots \otimes M_{j_{|w|}} e_{i_m} \otimes e_{i_1} \otimes \dots \otimes e_{i_{m-|w|}}. \end{aligned}$$

The right-hand side of this equality is a polynomial in scalar variables $(M_j)_{\alpha,\beta}$, $j = 1, \dots, N$; $\alpha, \beta = 1, \dots, m$, which vanishes identically. Thus, all of its coefficients are zeros. In particular,

$$\sum_{1 \leq i_1, \dots, i_m \leq m} P_{\emptyset} h_{i_1, \dots, i_m} \otimes e_{i_1} \otimes \dots \otimes e_{i_m} = 0.$$

Since the set of vectors $e_{i_1} \otimes \dots \otimes e_{i_m}$, $i_1, \dots, i_m \in \{1, \dots, m\}$, is linearly independent, $P_{\emptyset} h_{i_1, \dots, i_m} = 0$ for all $i_1, \dots, i_m \in \{1, \dots, m\}$. The coefficient for $(M_{j_1})_{1, \beta_1} \dots (M_{j_{|w|}})_{|w|, \beta_{|w|}}$ is

$$\sum_{\sigma} \sum_{1 \leq i_1, \dots, i_{m-|w|} \leq m} P_{g_{j_{\sigma(1)}} \dots g_{j_{\sigma(|w|)}}} h_{i_1, \dots, i_{m-|w|}, \beta_{\sigma(1)}, \dots, \beta_{\sigma(|w|)}} \otimes e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(|w|)} \otimes e_{i_1} \otimes \dots \otimes e_{i_{m-|w|}} = 0,$$

where σ runs over the set of all permutations of numbers $1, \dots, |w|$. Since the set of vectors $e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(|w|)} \otimes e_{i_1} \otimes \dots \otimes e_{i_{m-|w|}}$ is linearly independent, $P_{g_{j_{\sigma(1)}} \dots g_{j_{\sigma(|w|)}}} h_{i_1, \dots, i_{m-|w|}, \beta_{\sigma(1)}, \dots, \beta_{\sigma(|w|)}} = 0$ for all σ and $i_1, \dots, i_{m-|w|} \in \{1, \dots, m\}$. Since $\beta_1, \dots, \beta_{|w|} \in \{1, \dots, m\}$ are arbitrary, $P_w h_{i_1, \dots, i_m} = 0$ for all $w \in \mathbb{F}_N : |w| \leq m$, and $i_1, \dots, i_m \in \{1, \dots, m\}$. Thus, $h \in (\bigcap_{|w| \leq m} \ker P_w) \otimes \mathbb{C}^n$, as desired.

Now, let $m > 0$ and $n > m^m$. Consider any subspace V in \mathbb{C}^n of the form $V = \text{span}\{e_{j_1}, \dots, e_{j_m^m}\}$ where e_{j_k} are distinct vectors from the standard basis of \mathbb{C}^n . Define the set $\mathcal{M}_V \subset (\mathbb{C}^{n \times n})^N$ consisting of N -tuples of $n \times n$ matrices $Z = (Z_1, \dots, Z_N)$ such that $(Z_j)_{\alpha,\beta} = 0$ if $(\alpha, \beta) \notin \{j_1, \dots, j_m^m\} \times \{j_1, \dots, j_m^m\}$. Clearly, \mathcal{M}_V is isomorphic to $(\mathbb{C}^{m^m \times m^m})^N$. Thus,

$$\begin{aligned} \bigcap_{Z \in (\mathbb{C}^{n \times n})^N} \ker P(Z) &\subset \bigcap_{Z \in \mathcal{M}_V} \ker P(Z) = \left(\left(\bigcap_{w \in \mathbb{F}_N : |w| \leq m} \ker P_w \right) \otimes V \right) \oplus (\mathbb{C}^q \otimes (\mathbb{C}^n \ominus V)) \\ &= \left(\left(\bigcap_{w \in \mathbb{F}_N : |w| \leq m} \ker P_w \right) \otimes \mathbb{C}^n \right) \oplus \left(\left(\mathbb{C}^q \ominus \left(\bigcap_{w \in \mathbb{F}_N : |w| \leq m} \ker P_w \right) \right) \otimes (\mathbb{C}^n \ominus V) \right). \end{aligned}$$

Denote the latter orthogonal sum by $X_0 \oplus X_V$. Since $\bigcap_V X_V = \{0\}$ where V runs over the set of subspaces of the given form, we have $\bigcap_V (X_0 \oplus X_V) = X_0$. Therefore, $\bigcap_{Z \in (\mathbb{C}^{n \times n})^N} \ker P(Z) \subset (\bigcap_{|w| \leq m} \ker P_w) \otimes \mathbb{C}^n$.

The second part of this theorem follows from the fact that $\dim \bigcap_{Z \in (\mathbb{C}^{n \times n})^N} \ker P(Z) \leq qn$. The proof of Theorem 2.1 is complete.

Corollary 2.2. *Let $P(z) = \sum_{|w| \leq m} P_w z^w \in \mathbb{C}^{p \times q} \langle z_1, \dots, z_N \rangle$. If for some $n \in \mathbb{N}$: $n \geq m^m$ (in the case $m = 0$, for some $n \in \mathbb{N}$) one has $P(Z) = 0$, $\forall Z \in (\mathbb{C}^{n \times n})^N$, then $P(z) = 0$.*

Let us remark that our estimate m^m for a lower bound of n in Corollary 2.2 is, of course, much weaker than the estimate $[m/2] + 1$ well known in the theory of polynomial identities of rings (see p. 22 in [10] or pp. 22–23 in [11]). Nevertheless, it also implies that there is no polynomial relations valid for infinitely many matrix rings $\mathbb{C}^{n_j \times n_j}$, $n_j \in \mathbb{N}$, $j = 1, 2, \dots$. Our Theorem 2.1 implies the following generalization of the latter: there is no formal polynomial with matrix coefficients in general position (i.e., such that the common null space for all of them is zero), for which the common null space for all matrix substitutions into this polynomial is nonzero for an infinite number of matrix rings $\mathbb{C}^{n_j \times n_j}$, $n_j \in \mathbb{N}$, $j = 1, 2, \dots$.

Let us remark also that for any $\varepsilon > 0$ one can replace the set $(\mathbb{C}^{n \times n})^N$ in Theorem 2.1 by the set $\Gamma_n(\varepsilon)$ (see its definition in Section 1). Indeed, by the uniqueness theorem, if $P(Z)x = 0$ for some $x \in \mathbb{C}^q \otimes \mathbb{C}^n$ and all $Z \in \Gamma_n(\varepsilon)$ then $P(Z)x = 0$ for all $Z \in (\mathbb{C}^{n \times n})^N$.

3. The general case

Theorem 3.1. Let $F(z) = \sum_{w \in \mathbb{F}_N} F_w z^w \in \mathbb{C}^{p \times q} \langle \{z_1, \dots, z_N\} \rangle_{\text{rat}}$, and $m \in \mathbb{Z}_+$ be such that

$$\bigcap_{w \in \mathbb{F}_N: |w| \leq m} \ker F_w = \bigcap_{w \in \mathbb{F}_N} \ker F_w.$$

Then there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$: $n \geq m^m$ (in the case $m = 0$, for every $n \in \mathbb{N}$),

$$\bigcap_{Z \in \Gamma_n(\varepsilon)} \ker F(Z) = \left(\bigcap_{w \in \mathbb{F}_N: |w| \leq m} \ker F_w \right) \otimes \mathbb{C}^n, \tag{3}$$

and moreover, there exist $l \in \mathbb{N}$: $l \leq qn$, and N -tuples of matrices $Z^{(1)}, \dots, Z^{(l)} \in \Gamma_n(\varepsilon)$ such that

$$\bigcap_{j=1}^l \ker F(Z^{(j)}) = \left(\bigcap_{w \in \mathbb{F}_N: |w| \leq m} \ker F_w \right) \otimes \mathbb{C}^n.$$

Proof. Recall (see Section 1) that there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ the function $F(Z)$, which is considered as a function of scalar variables $(Z_k)_{ij}$, $k = 1, \dots, N$, $i, j = 1, \dots, n$, is holomorphic on $\Gamma_n(\varepsilon)$. If $m = 0$ then equality (3) is obvious for any $n \in \mathbb{N}$. If $m > 0$ then the inclusion “ \supset ” in (3) is also obvious for any $n \in \mathbb{N}$. Conversely, if $n \geq m^m$ and $x \in \ker F(Z)$, $\forall Z \in \Gamma_n(\varepsilon)$, then $P_k(Z)x \equiv 0$ for every homogeneous polynomial $P_k(z)$ in the expansion of $F(z)$ in homogeneous polynomials. Therefore, $x \in \ker P(Z)$, $\forall Z \in \Gamma_n(\varepsilon)$, where $P(z) = \sum_{|w| \leq m} F_w z^w$. By Theorem 2.1 and the remark in the last paragraph of Section 2, $x \in (\bigcap_{|w| \leq m} \ker F_w) \otimes \mathbb{C}^n$, and the inclusion “ \subset ” in (3) follows. The second part of this theorem follows in the same way as in Theorem 2.1. The proof of Theorem 3.1 is complete.

Corollary 3.2. In conditions of Theorem 3.1, if for some $n \in \mathbb{N}$: $n \geq m^m$ (in the case $m = 0$, for some $n \in \mathbb{N}$) one has $F(Z) = 0$, $\forall Z \in \Gamma_n(\varepsilon)$, then $F(z) = 0$.

Thus, there is no rational identities valid for infinitely many matrix rings $\mathbb{C}^{n_j \times n_j}$, $n_j \in \mathbb{N}$, $j = 1, 2, \dots$, and our Theorem 3.1 can be considered as a generalization of this fact. Note that by “rational” we mean here rational formal power series, however the latter fact is valid also in the general case of rational expressions in non-commuting indeterminates, which follows from the theory of rational identities of division rings (see Section 8.2 of [11]).

Recall (see, e.g., [2]) that a representation (1) of a rational formal power series $F(z)$ is called observable (resp., controllable) if the infinite matrix $\mathcal{O}_F = \text{col}(CA^w)_{w \in \mathbb{F}_N}$ (resp., $\mathcal{C}_F = \text{row}(A^{w^T}B)_{w \in \mathbb{F}_N}$) is of full column (resp., row) rank (= r). This representation is minimal, i.e., $r = \text{rank } \mathcal{H}_F$ (see Section 1), if and only if it is observable and controllable, since $\mathcal{H}_F = \mathcal{O}_F \mathcal{C}_F$.

Theorem 3.3. A representation (1) of $F(z) \in \mathbb{C}^{p \times q} \langle \{z_1, \dots, z_N\} \rangle_{\text{rat}}$, with $C \in \mathbb{C}^{p \times r}$, $A = (A_1, \dots, A_N) \in (\mathbb{C}^{r \times r})^N$, $B \in \mathbb{C}^{r \times q}$, is observable (resp., controllable) if and only if for every $n \in \mathbb{N}$ such that $n \geq (pr - 1)^{pr-1}$ (resp., $n \geq (rq - 1)^{rq-1}$), and for every $n \in \mathbb{N}$ in the case $pr = 1$ (resp., $rq = 1$):

$$\bigcap_{Z \in \Gamma_n(\varepsilon)} \ker \left\{ (C \otimes I_n) \left(I_r \otimes I_n - \sum_{k=1}^N A_k \otimes Z_k \right)^{-1} \right\} = 0 \tag{4}$$

$$\left(\text{resp., } \bigvee_{Z \in \Gamma_n(\varepsilon)} \text{range} \left\{ \left(I_r \otimes I_n - \sum_{k=1}^N A_k \otimes Z_k \right)^{-1} (B \otimes I_n) \right\} = \mathbb{C}^r \otimes \mathbb{C}^n \right), \tag{5}$$

where $\varepsilon = (N \max_{1 \leq k \leq N} \|A_k\|)^{-1}$ ($\varepsilon > 0$ is arbitrary in the case $A = 0$), and “ $\bigvee_j X_j$ ” denotes the linear span of sets X_j . This representation is minimal if both of conditions (4) and (5) are fulfilled.

Proof. Let $d \in \mathbb{Z}_+$ be the least number such that all of matrices CA^w , $|w| = d + 1$, are linearly dependent on matrices $CA^{w'}$, $|w'| \leq d$. Then for any $w \in \mathbb{F}_N$: $|w| = d + 1$, and $k \in \{1, \dots, N\}$ one has

$$CA^{wgk} = CA^w A_k = \sum_{|w'| \leq d} \alpha_{w'} CA^{w'} A_k = \sum_{|w'| < d} \alpha_{w'} CA^{w'gk} + \sum_{|w'| = d} \sum_{w'' \leq d} \alpha_{w'} \beta_{w', w''} CA^{w''},$$

with some $\alpha_{w'}, \beta_{w', w''} \in \mathbb{C}$. Thus, every matrix CA^w with $|w| = d + 2$ is linearly dependent on matrices $CA^{w'}$, $|w'| \leq d$. By induction, this is also true for every matrix CA^w with $|w| = d + 3, \dots$. Since matrices CA^w are of size $p \times r$, we have $d \geq pr - 1$. Therefore, $\text{rank}\{\text{col}(CA^w)_{|w| \leq pr-1}\} = \text{rank } \mathcal{O}_F$. Applying Theorem 3.1 to the rational formal power series $\Phi(z) = \sum_{w \in \mathbb{F}_N} CA^w z^w = C(I_r - \sum_{k=1}^N A_k z_k)^{-1}$ in the place of $F(z)$, and $m = pr - 1$, we obtain the statement on the observability of representation (1). The statement on the controllability of representation (1) is proved analogously. The statement on minimality now is obvious. The proof of Theorem 3.3 is complete.

We can obtain similar criteria for various types of non-commutative multidimensional systems which appear, e.g., in [3,9]. Such criteria for linear and non-linear system realizations of non-commutative rational formal power series since works of Fliess (e.g., see [5,6]) were formulated in terms of coefficients of a series or system's matrix. Our criteria are given in terms of values of a series on matrices, which reduces a problem to analysis of rational functions of several commuting variables (see Introduction).

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