Available online at www.sciencedirect.com
SCIENCE DIRECT
C. R. Acad. Sci. Paris, Ser. I 339 (2004) 483–486
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

Sharp Hardy–Sobolev inequalities

S. Filippas [a,b], V.G. Maz'ya [c,d], A. Tertikas [b,e]

[a] *Department of Applied Mathematics, University of Crete, 71409 Heraklion, Greece*
[b] *Institute of Applied and Computational Mathematics FORTH, 71110 Heraklion, Greece*
[c] *Department of Mathematics, Ohio State University, Columbus, OH 43210, USA*
[d] *Department of Mathematics, Linkoeping University, 58183 Linkoeping, Sweden*
[e] *Department of Mathematics, University of Crete, 71409 Heraklion, Greece*

Received and accepted 26 July 2004

Available online 25 September 2004

Presented by Haïm Brezis

Abstract

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 3$. We show that Hardy's inequality involving the distance to the boundary, with best constant $(1/4)$, may still be improved by adding a multiple of the critical Sobolev norm. ***To cite this article: S. Filippas et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Inégalités de Hardy–Sobolev précisées. Soit Ω un ouvert borné et regulier dans $\mathbb{R}^N$, $N \geqslant 3$. On montre que l'inegalité de Hardy, liée à la distance au bord, avec meilleure constante $(1/4)$, peut être améliorée en ajoutant un multiple de la norme de Sobolev critique. ***Pour citer cet article : S. Filippas et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction and main results

If $K = \{x \in \mathbb{R}^N \mid x_1 = x_2 = \cdots = x_k = 0\}$, $1 \leqslant k \leqslant N-1$, and $d(x) = \operatorname{dist}(x, K)$ the following Hardy–Sobolev inequality with critical exponent has been established in ([6], Corollary 3, p. 97)

$$\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{d^2} \,\mathrm{d}x \geqslant C\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \,\mathrm{d}x\right)^{\frac{N-2}{N}}, \quad \forall u \in C_0^\infty(\mathbb{R}^N \setminus K). \tag{1}$$

When $k = N$, then $K = \{0\}$, $d(x) = |x|$ and (1) fails. To state the analogue inequality in this case, let $X(r) := (1 - \ln r)^{-1}$, $0 < r \leqslant 1$. We also set $D := \sup_{x \in \Omega} |x|$. Then for any bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$ there holds

E-mail addresses: filippas@tem.uoc.gr (S. Filippas), vlmaz@mai.liu.se (V.G. Maz'ya), tertikas@math.uoc.gr (A. Tertikas).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.07.023

([5], Theorem A)

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\,\mathrm{d}x \geqslant C\left(\int_\Omega |u|^{\frac{2N}{N-2}} X^{\frac{2(N-1)}{N-2}}\left(\frac{|x|}{D}\right)\mathrm{d}x\right)^{\frac{N-2}{N}}, \quad \forall u \in C_0^\infty(\Omega). \tag{2}$$

Inequality (2) involves the critical exponent, but contrary to (1) it has a logarithmic correction. Moreover, it is sharp in the sense that one cannot take a smaller power of the logarithmic correction X.

We next present recent results that extent (1) to more general domains Ω and distance functions. To simplify the presentation from now on we consider only the case where $K = \partial\Omega$, and therefore $d(x) = \operatorname{dist}(x, \partial\Omega)$, we emphasize however that all the results that follow have a counterpart in the case where K is a smooth manifold of codimension k, with $1 \leqslant k \leqslant N-1$.

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a smooth and convex domain with $D := \sup_{x\in\Omega} d(x) < \infty$. Then the following inequality is true ([1], Theorem 6.4)

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,\mathrm{d}x \geqslant C\left(\int_\Omega |u|^{\frac{2N}{N-2}} X^{\frac{2N}{N-2}}\left(\frac{d(x)}{D}\right)\mathrm{d}x\right)^{\frac{N-2}{N}}, \quad \forall u \in C_0^\infty(\Omega). \tag{3}$$

On the other hand if Ω is a bounded smooth domain (no convexity is required) the following inequality has been proved by Dávila and Dupaigne ([3], Theorem 1)

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,\mathrm{d}x + \lambda\int_\Omega u^2\,\mathrm{d}x \geqslant C\left(\int_\Omega |u|^q\,\mathrm{d}x\right)^{\frac{2}{q}}, \quad \forall u \in C_0^\infty(\Omega), \tag{4}$$

for λ and C positive constants depending on Ω, and $1 \leqslant q < q_1 := \frac{2(N+1)}{N-1}$.

Inequality (3) requires convexity of Ω and misses the critical exponent by a logarithmic correction. On the other hand no convexity is needed for (4) at the expense of adding an L^2 norm in the left-hand side and staying below the exponent q_1 ($< \frac{2N}{N-2}$) in the right-hand side.

In this work we improve both (3) and (4) by obtaining the sharp analogue of (1). Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 3$, $d(x) = \operatorname{dist}(x, \partial\Omega)$ and let $\Omega_\delta := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \leqslant \delta\}$ be a tubular neighborhood of $\partial\Omega$. We then have

Theorem 1.1. *There exist positive constants $\lambda = \lambda(\Omega)$ and $C = C(\Omega)$ depending on Ω such that*

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,\mathrm{d}x + \lambda\int_\Omega u^2\,\mathrm{d}x \geqslant C\left(\int_\Omega |u|^{\frac{2N}{N-2}}\,\mathrm{d}x\right)^{\frac{N-2}{N}}, \quad \forall u \in C_0^\infty(\Omega). \tag{5}$$

No convexity of Ω is needed. We note that the first Hardy-type result that dismisses convexity at the expense of adding a lower order term, is the following inequality due to Brezis and Marcus ([2], Theorem I)

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,\mathrm{d}x + \lambda\int_\Omega u^2\,\mathrm{d}x \geqslant 0, \quad \forall u \in C_0^\infty(\Omega), \tag{6}$$

for a constant λ that depends on Ω. In case Ω is convex we have

Theorem 1.2. *If Ω is convex, there exists a positive constant $C = C(\Omega)$ depending on Ω such that*

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,\mathrm{d}x \geqslant C\left(\int_\Omega |u|^{\frac{2N}{N-2}}\,\mathrm{d}x\right)^{\frac{N-2}{N}}, \quad \forall u \in C_0^\infty(\Omega). \tag{7}$$

Inequality (7) is scale invariant and the constant C we have computed depends on Ω in a scale invariant way. The following then is a natural question.

Open problem 1: Are the constants C of Theorems 1.1 and 1.2 independent of Ω?

The results that follow strongly suggest that C is independent of Ω.

Theorem 1.3. *There exists a positive constant $\delta_0 = \delta_0(\Omega)$ depending on Ω and a positive constant $C = C(N)$ depending only on the dimension N, such that for all $0 < \delta \leqslant \delta_0$*

$$\int_{\Omega_\delta} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{4}\int_{\Omega_\delta} \frac{u^2}{d^2}\,\mathrm{d}x \geqslant C\biggl(\int_{\Omega_\delta} |u|^{\frac{2N}{N-2}}\,\mathrm{d}x\biggr)^{\frac{N-2}{N}}, \quad \forall u \in C_0^\infty(\Omega_\delta). \tag{8}$$

Notice that Ω need not be convex for Theorem 1.3. To state our next result we introduce some notation. We denote by $\mathcal{C}_\phi$ or simply by $\mathcal{C}$ the circular cone with vertex at the origin and having axis of symmetry the positive x_N-axis. The angle $\phi \in (0, \frac{\pi}{2})$ is the angle between any line on $\mathcal{C}$ passing through the origin and the positive x_N-axis. Let $d(x) = \mathrm{dist}(x, \partial\mathcal{C})$. We then have

Theorem 1.4. *There exists a constant $C = C(N)$ depending only on the dimension N such that*

$$\int_{\mathcal{C}} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{4}\int_{\mathcal{C}} \frac{|u|^2}{d^2}\,\mathrm{d}x \geqslant C\biggl(\int_{\mathcal{C}} |u|^{\frac{2N}{N-2}}\,\mathrm{d}x\biggr)^{\frac{N-2}{N}}, \quad \forall u \in C_0^\infty(\mathcal{C}). \tag{9}$$

We note in particular that the constant C in (9) is independent of the opening angle ϕ of the cone for $\phi \in (0, \frac{\pi}{2})$. The last theorem also raises the question of what are the minimal assumptions on Ω – besides convexity – under which (7) remains valid.

It will be interesting to compute the best constant C at least for special geometries.

Open problem 2: What is the best constant C for the unit ball or for half space? How does it relate to the best Sobolev constant?

Remark 1. Theorems 1.1–1.3 have a counterpart in the case where K is a smooth manifold of codimension k with $1 < k \leqslant N-1$ and $d(x) = \mathrm{dist}(x, K)$. In all these cases the critical Sobolev norm appears in the right-hand side. We also note that most of the results extend to the L^p setting for $2 \leqslant p < N$.

2. Sketch of proofs

We first present the key ingredients in the proof of Theorems 1.2 and 1.3. By the change of variables $u(x) = d^{1/2}(x)v(x)$ (cf. [1]) inequality (7) is equivalent to

$$\int_{\Omega} d|\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\Omega} (-\Delta d)|v|^2\,\mathrm{d}x \geqslant C\biggl(\int_{\Omega} d^{\frac{N}{N-2}}|v|^{\frac{2N}{N-2}}\,\mathrm{d}x\biggr)^{\frac{N-2}{N}}, \quad \forall v \in C_0^\infty(\Omega), \tag{10}$$

valid for $v \in C_0^\infty(\Omega)$. To prove (10) we will derive suitable inequalities near the boundary as well as away from the boundary and then we will combine them. To this end let ϕ_δ be a smooth cutoff such that $\phi_\delta = 1$ in Ω_δ and $\phi_\delta = 0$ in $\Omega_{2\delta}^C$ and set $v = \phi_\delta v + (1-\phi_\delta)v =: v_1 + v_2$. Near the boundary, that is, for δ small enough, and for smooth domains, $d(x)$ is a smooth function and Δd approaches the mean curvature of the boundary. As a consequence, the middle integral in (10) is treated as a lower order term. The desired estimate then for v_1 follows by the Gagliardo–Nirenberg inequality and elementary estimates. This proves Theorem 1.3, and no convexity is needed.

To prove Theorem 1.2, we need in addition to work with v_2. We note that away from the boundary, $\delta \leqslant d(x) \leqslant D = \sup_{x\in\Omega} d(x)$, and therefore (10) is easily seen to be true for v_2, even if the term containing $-\Delta d$ is absent.

Convexity is now needed to guarantee that $-\Delta d \geqslant 0$. We finally note that the dependence of C on Ω enters through the ratio $\frac{\delta}{D}$ – a scale invariant quantity.

Proof of Theorem 1.1. Let ϕ_δ be a smooth cutoff such that $\phi_\delta = 1$ in Ω_δ and $\phi_\delta = 0$ in $\Omega_{2\delta}^C$. We write $u = \phi_\delta u + (1-\phi_\delta)u =: u_1 + u_2$. We then follow closely the argument of ([7], Theorem 2.2, case 3), or ([3], Theorem 1). That is, by a straightforward calculation we have that

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,\mathrm{d}x = \int_\Omega |\nabla u_1|^2\,\mathrm{d}x - \frac{1}{4}\int_\Omega \frac{u_1^2}{d^2}\,\mathrm{d}x + \int_\Omega |\nabla u_2|^2\,\mathrm{d}x + R(u_1,u_2), \tag{11}$$

for a suitable remainder term R. The term R is easily estimated from below by $-\lambda\int_\Omega u^2\,\mathrm{d}x$. The first two terms of the right-hand side are estimated by Theorem 1.1, whereas for the gradient term we use the standard Sobolev imbedding and the result follows. □

Proof of Theorem 1.4. We will use the self similarity of the cone $\mathcal{C}$. We denote by $\mathcal{C}_{(1,2)}$ the intersection of $\mathcal{C}$ with the strip $\mathbb{R}^{N-1} \times (1 < x_N < 2)$ and we notice that $\mathcal{C} = \bigcup_{n=-\infty}^{\infty} 2^n \mathcal{C}_{(1,2)}$. It is enough to prove inequality (9) for $\mathcal{C}_{(1,2)}$ when distance is taken *only* from the lateral surface of $\mathcal{C}_{(1,2)}$. By scale invariance then, the inequality is true for all pieces $2^n\mathcal{C}_{(1,2)}$, $n = \pm 1, \pm 2, \ldots,$ with the same constant, and we can patch them together to get (9).

By practically the same argument as in the proof of Theorem 1.2, we can obtain inequality (7) for the unit cylinder $H = B_1^{N-1} \times (0,1)$, when distance is taken *only* from the lateral surface $L = \partial B_1^{N-1} \times (0,1)$. It is easy then to map H onto $\mathcal{C}_{(1,2)}$ in a one-to-one way by an elementary transformation which is bi-Lipschitz. This will give the inequality for $\mathcal{C}_{(1,2)}$ with a constant C that has a positive limit as ϕ tends to zero, but unfortunately we loose control of the constant C as ϕ tends to $\frac{\pi}{2}$. This is in a sense expected since we 'perturbed' the cylinder to get the cone and when $\phi = \frac{\pi}{2}$ we are in the case of half space.

For the half space however the inequality is true (see (1) with $k = 1$). We then use a similar argument, that is, we map the half space with a bi-Lipschitz elementary transformation onto the cone for, say, $\phi \in (\frac{\pi}{4}, \frac{\pi}{2})$, and this eventually shows that the constant C stays away from zero or infinity even in the case where ϕ tends to $\frac{\pi}{2}$. The result then follows easily.

Remark 2. Detailed proofs as well as various extensions of the results we presented here, will be given in a forthcoming publication (cf. [4]).

Acknowledgements

This work started when VGM visited the University of Crete, the hospitality and support of which is acknowledged. SF and AT acknowledge partial support by the RTN European network Fronts–Singularities, HPRN-CT-2002-00274. SF acknowledges additional support by RTN European network HYKE, HPRN-CT-2002-00282.

References

[1] G. Barbatis, S. Filippas, A. Tertikas, A unified approach to improved L^p Hardy inequalities with best constants, Trans. Amer. Math. Soc. 356 (6) (2004) 2169–2196.
[2] H. Brezis, M. Marcus, Hardy's inequalities revisited, Ann. Scuola Norm. Pisa 25 (1997) 217–237.
[3] J. Dávila, L. Dupaigne, Hardy-type inequalities, J. Eur. Math. Soc. 6 (3) (2004) 335–365.
[4] S. Filippas, V.G. Maz'ya, A. Tertikas, Critical Hardy Sobolev inequalities, in preparation.
[5] S. Filippas, A. Tertikas, Optimizing improved Hardy inequalities, J. Funct. Anal. 192 (2002) 186–233.
[6] V.G. Maz'ya, Sobolev Spaces, Springer, 1985.
[7] J.L. Vázquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal. 173 (2000) 103–153.