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Dynamical Systems

Thickness of Julia sets of Feigenbaum polynomials with high order critical points

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Abstract

We consider unimodal polynomials with Feigenbaum topological type and critical points whose orders tend to infinity. It is shown that the hyperbolic dimensions of their Julia set go to 2; furthermore, that the Hausdorff dimensions of the basins of attraction of their Feigenbaum attractors also tend to 2. The proof is based on constructing a limiting dynamics with a flat critical point. ***To cite this article: G. Levin, G. Świątek, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

L'épaisseur des ensembles de Julia de polynômes de Feigenbaum ayant des points critiques d'ordre élevé. On considère des polynômes unimodaux de type topologique de Feigenbaum et les points critiques dont l'ordre tend vers l'infini. On montre que la dimension hyperbolique des ensembles de Julia tend vers 2. De plus, la dimension de Hausdorff du bassin d'attraction des attracteurs tend aussi vers 2. La preuve s'appuie sur une construction de la dynamique limite avec un point critique plat.
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1. Main results

A significant problem in holomorphic dynamics concerns the sizes of Julia set of rational maps. Not much is known about this problem in general except for the hyperbolic case. A principle which has recently emerged is

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that a certain amount of expansion on the post-critical set favors a thin Julia set, while the lack of such expansion will make the Julia set thick. In the former direction, one can mention results obtained under Collet–Eckmann type of conditions, see [3] and [6]. In the other direction, one has the results of [7] which has found many Julia set of Hausdorff dimension 2 for quadratic polynomials and [1], which is not directly related to holomorphic dynamics, but appears indicative of what one might expect to find. The lack of expansion in these cases may be due to parabolic phenomena or to a highly degenerate form of the critical point itself.

Based on this principle, for infinitely renormalizable polynomials one might expect to find thick Julia sets, particularly when the critical point is very degenerate. We provide some evidence of that by showing that the Hausdorff dimension of Julia sets of unimodal polynomials tends to 2 as the degree of degeneracy of their critical points grows, see Theorem 1.1. Our method is based on constructing a limiting map for which the critical point is infinitely degenerate (flat) and obtaining estimates by viewing the high degree polynomials as perturbations of the limit map.

Notations and basic facts

We will write unimodal mappings of an interval, $H\colon [0,1] \to [0,1]$ in the following non-standard form $H(x) = |E(x)|^{\ell}$, where $\ell > 1$ is a real number and E is a analytic mapping with strictly negative derivative on [0, 1] which maps 0 to 1 and 1 to a point inside $(-1,0)$. Then H is unimodal with the minimum at some $x_0 = E^{-1}(0) \in (0,1)$ and x_0 is the critical point of order ℓ.

For every $\ell > 1$ even integer there exists a unique fixed point $H_\ell(x) = |E_\ell(x)|^{\ell}$ of the *Feigenbaum functional equation*

$$\tau H^2(x) = H(\tau x) \tag{1}$$

for $x \in [0, \tau^{-1}]$ with $\tau := \tau_\ell > 1$.

Mappings H_ℓ, ℓ even, belong to the *Epstein class*, i.e. whenever $H_\ell(z) = z_1 \notin \mathbb{R}$, then there is an inverse branch of H_ℓ defined on the upper or lower half plane, depending on the position of z_1, which maps z_1 to z.

The main results of this paper are contained in the following theorem:

Theorem 1.1. *For every $\epsilon > 0$ there is ℓ_ϵ such that whenever G is a real unimodal polynomial-like mapping which is conjugated to the Feigenbaum map and all derivatives of G of order less than ℓ_ϵ vanish at its critical point, then*:

- *the filled-in Julia set of G contains a hyperbolic invariant subset with Hausdorff dimension greater than $2-\epsilon$,*
- *the set of points of the filled-in Julia set of G whose ω-limit set is equal to the ω-limit set of the critical point has Hausdorff dimension greater than $2-\epsilon$.*

For Theorem 1.1 our non-standard normalization of unimodal map does not make a difference, although it makes an important difference in the proof.

2. Limits as $\ell \to \infty$

The proof of Theorem 1.1 is based on an understanding of the metric dynamics of a limit map as ℓ tends to ∞ and in particular on the construction of scaling-invariant Markov partition. Then the claims of the theorem are derived by perturbing that Markov partition.

As ℓ goes to ∞, mappings H_ℓ converge to a non-trivial analytic limit. This is true owing to the normalization we use. Details were studied in [2,4]. Let us list the facts relevant for our problem:

– On the interval [0, 1], H_ℓ converge uniformly to a unimodal map H with a critical point at x_0 which satisfies the Feigenbaum fixed point equation (1).

– H has an analytic continuation to the union of two topological disks $\Omega_- \ni 0$ and Ω_+, both symmetric with respect to the real axis with closures intersecting exactly at x_0. The boundaries of Ω_+ and Ω_- are Jordan curves with Hausdorff dimension 1.
– On any compact subset of $\Omega_+ \cup \Omega_-$, H_ℓ are defined and analytic for all ℓ large enough and converge uniformly to H.

Let us denote $H_k(z) = \tau^k H(z\tau^{-k})$ for any integer k. Because of the fixed point equation, $H_k^2 = H_{k-1}$. Also, we introduce an annulus $B := \Omega_- \setminus \tau^{-1}\overline{\Omega_-}$. Let B' denote the image of B under the principal branch of the log. Hence, B' by definition is contained in the strip $\{z\colon |\Im z| < \pi\}$.

The key construction is described by the following theorem:

Theorem 2.1. *For a fixed open neighborhood C' of $\overline{B'}$ and for every choice of $\alpha > 0$ and integer κ, there are an integer N, constant $C_1 > 0$ and a map $\Gamma_{\alpha,\kappa}$ which is induced by H_N on the union of finitely many topological disks V_i, all contained with their closures inside B. These dynamical properties hold*:

– *$\Gamma_{\alpha,\kappa} = H_N^{j_i}$ maps each V_i onto $\tau^\kappa(B) \setminus (-\infty, 0]$. Furthermore, the entire trajectory $V_i, H_N(V_i), \ldots H_N^{j_i}(V_i)$ is contained in $\tau^N(B)$ and avoids $\tau^{-N-1}(B)$.*
– *for every i, if $\Gamma'_{i,\alpha,\kappa} : B' \to B' + \kappa \log\tau$ denotes the mapping $\log(\Gamma_{\alpha,\kappa}(\exp(z)))$ restricted to $\log(V_i)$, then $\Gamma'_{i,\alpha,\kappa}$ continues analytically to a univalent map defined on a set compactly contained in C' onto C'.*

To state additional metric *properties denote $M_{\alpha,\kappa} := \max\{\operatorname{diam} V_i\}$, $m_{\alpha,\kappa} = \min\{\operatorname{diam} V_i\}$ and $\mu_{\alpha,\kappa}$ the joint measure of sets V_i. Then*

$$\log \mu_{\alpha,\kappa}^{-1} < \alpha \log M_{\alpha,\kappa}^{-1}, \qquad \log m_{\alpha,\kappa}^{-1} < C_1 \log M_{\alpha,\kappa}^{-1}.$$

Observe that the extension condition in Theorem 2.1 means that all iterates of the map $\tau^{-\kappa}\Gamma_{\alpha,\kappa}$ have uniformly bounded distortion, which only depends on the geometry of the set C and is therefore independent from α and κ.

The proof of Theorem 2.1 is based first on constructing a dynamics on B:

Lemma 2.2. *For every N_0 there is a mapping Γ_{N_0} induced by H_{N_0} which is defined on the union of topological disks W_i each of which is mapped onto $\tau^k B \setminus (-\infty, 0]$ for some $k = N_0 + 1, N_0, \ldots, 0, -1, \ldots$. The map Γ_{N_0} satisfies the extension and dynamical properties as in Theorem* 2.1.

The metric properties are:

– *the measure of $B \setminus \bigcup W_i$ tends to 0 as N_0 increases to ∞,*
– *there exist positive constants K_1, K_2 such that for every $k \leqslant N_0$, the joint measure of those W_i which are mapped onto $\tau^k B \setminus (-\infty, 0]$ is at least $K_1|k|^{-K_2}$, in other words polynomial in terms of k.*

The last statement is crucial and it follows from the form of the flat critical point of H.

To continue the construction using Γ_{N_0}, one further eliminates some of the disks W_i which may be too small in diameter or are mapped onto $\tau^k B \setminus (-\infty, 0]$ for k too large. The remaining W_i are only finitely many, and each of them maps onto $\tau^k B \setminus (-\infty, 0]$ with k between $-N_0$ and N_0. The smaller the α requested in Theorem 2.1, the less measure can be eliminated which means that larger N_0 and smaller diameters need to be allowed. The map Γ_{N_0} can be iterated in the sense that if W_i is mapped onto $\tau^k B \setminus (-\infty, 0]$, then $\Gamma_{N_0}(\tau^{-k}\Gamma_{N_0})$ makes sense on all W_j, etc. This iteration goes on for k_0 steps, leading to an exponential shrinking of the diameters of the pieces and an exponential loss of the measure of points which are still in play, but the ratio of that loss can be made arbitrarily close to 1. The images of those pieces now inhabit various scales between $\tau^{-k_0 N_0}$ and $\tau^{k_0 N_0}$.

At that point a 'forcing step' which uses $\Gamma_{k_0 N_0}$ is employed in which all pieces which fail to map to the desired scale $\tau^\kappa B \setminus (-\infty, 0]$ are eliminated. This of course leads to a very significant loss of measure to discarded pieces. However, in view of the last claim of Lemma 2.2, then portion which remains is sub-exponentially small in terms of k_0. Thus, if k_0 is chosen sufficiently large depending on N_0, the average exponential rate of decay of the measure per one step is essentially unaffected by the forcing step and can remain very close to 1.

In this way the metric estimates of Theorem 2.1 are satisfied, while the dynamical properties follow directly from the features of the limit map.

Estimates of Hausdorff dimension

These estimates can be obtained using Frostman's Lemma and the result is:

Lemma 2.3. *If a mapping $\Gamma_{\alpha,\kappa}$ satisfies the claim of Theorem* 2.1, *regardless of whether it is induced by any ambient dynamics and for an arbitrary annulus B surrounding* 0, *then the Hausdorff dimension of the set of points which remain in the domain of Γ forever under the iteration by $\tau^{-\kappa}\Gamma_{\alpha,\kappa}$ has Hausdorff dimension as least* $2 - 2\alpha$.

When $\kappa = 0$, then points which can be forever iterated by $\Gamma_{\alpha,\kappa}$ are in fact forever in the domain of H_N and avoid $\tau^{-N-k}B$ for all $k > 0$. It follows that H_N is uniformly expanding on the union of images of this set. Thus, in the light of Lemma 2.3, the hyperbolic Hausdorff dimension of the Julia set of H_N, and hence of H, is at least $2 - 2\alpha$.

When $\kappa = -1$ points which can forever be iterated by $\tau\Gamma_{\alpha,\kappa}$ are also in the filled-in Julia set of H_N. Moreover, $\Gamma_{\alpha,-1}$ maps such points into the filled-in Julia set of H_{N-1}. Since H has the Epstein property, this implies that the ω-limit set of such points is the Feigenbaum attractor.

In this way, we conclude that for the limiting map the Hausdorff dimension of the set of points whose ω-limit set coincides with the Feigenbaum attractor as well as the hyperbolic dimension of Julia set are equal to 2. By slightly different considerations also based on the induced dynamics and with improved control of the distortion, the area of the Julia set is zero.

3. Derivation of main results

A proof of Theorem 1.1 is obtained by perturbing the construction of the map $\Gamma_{\alpha,\kappa}$. Recall that H_ℓ is the fixed point of Feigenbaum's equation (1) with the critical point degenerate of order ℓ and denote $H_{\ell,k}(x) := \tau_\ell^k H_\ell(\tau_\ell^{-k} x)$. Using the convergence, we easily observe that for each choice of α, κ, $\Gamma_{\ell,\alpha,\kappa}$ with the properties listed in the claim of Theorem 2.1 can still be induced by $H_{\ell,N}$ instead of H_N provided that order of degeneracy ℓ be large enough depending on α and κ.

Once $\Gamma_{\ell,\alpha,\kappa}$ has been obtained, Lemma 2.3 can be used again. This yields the proof of Theorem 1.1 for maps H_ℓ. Finally, the results generalize to all polynomial-like maps by convergence of renormalization, [5].

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