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Partial Differential Equations

Monotone approximations of Green's functions

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Abstract

We study the approximations of the Green's function $\mathbb{G}$ in a domain Ω obtained from an approximation of the Dirac mass δ_0. We prove that under some conditions, these approximations converge monotonically to $\mathbb{G}$, a rather surprising result. ***To cite this article: E. Chasseigne, R. Ferreira, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Approximation monotone des fonctions de Green. Nous étudions les approximations des fonctions de Green $\mathbb{G}$ dans un domaine Ω obtenues par approximation de la masse de Dirac δ_0. Nous montrons que sous certaines conditions, ces approximations sont monotones, ce qui peut paraître surprenant. ***Pour citer cet article : E. Chasseigne, R. Ferreira, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Soit Ω un ouvert régulier borné de $\mathbb{R}^d$, $d \geqslant 2$, contenant l'origine. On note $\mathbb{G}(x, y)$ la fonction de Green du Laplacien (on trouvera une étude complète sur ce sujet dans Bénilan [1]) dans Ω, c'est-à-dire la solution du problème :

$$\begin{cases} -\Delta\mathbb{G}(x,\cdot) = \delta_x(\cdot) & \text{dans } \Omega, \\ \mathbb{G}(x,\cdot) = 0 & \text{sur } \partial\Omega, \end{cases} \tag{1}$$

le point x étant fixé dans Ω. Dans le cas $x = 0$, on notera $G(\cdot) = \mathbb{G}(0, \cdot)$.

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Une méthode classique pour construire G consiste à approcher la masse de Dirac δ_0 par une suite $\rho = (\rho_n)$ de fonctions régulières convergeant faiblement vers δ_0 et résoudre le problème suivant : $-\Delta G_n(x) = \rho_n(x)$, avec $G_n = 0$ sur le bord. Il est bien connu qu'il existe une unique solution G_n de ce problème et que lorsque n tend vers l'infini, la suite (G_n) converge vers la fonction G. Dans cette Note, nous montrons que la convergence est monotone si certaines conditions sur la suite (ρ_n) sont satisfaites.

Définition 0.1. On dira que la suite $\rho = (\rho_n)$ est une bonne approximation de la masse de Dirac δ_0 dans Ω si les conditions suivantes sont remplies :

(i) $\rho_n \in C^0(\mathbb{R}^d)$, $\rho_n \geqslant 0$, $\rho_n(x) = \rho_n(|x|)$ est radiale, et $\operatorname{supp}(\rho_n) \subset \Omega$,
(ii) $\int_\Omega \rho_n(x)\,dx = 1$, $\int_K \rho_n(x)\,dx \to 0$ pour tout compact $K \subset \Omega \setminus \{0\}$,
(iii) pour tout $n \in \mathbb{N}$, il existe un unique $\eta_n \in \mathbb{R}$ tel que $B(\eta_n) \subset \Omega$ et

$$\begin{cases} \rho_n(x) < \rho_{n+1}(x) & \text{si } 0 \leqslant |x| < \eta_n, \\ \rho_n(x) > \rho_{n+1}(x) & \text{si } |x| > \eta_n. \end{cases} \tag{2}$$

Nous prouvons alors le

Théorème 0.2. *Soit ρ une bonne approximation de δ_0 dans Ω. Alors* :

(i) *La suite (G_n) est croissante et converge vers G dans Ω.*
(ii) *Si $\Omega \setminus \operatorname{supp}(\rho_n) \neq \emptyset$, alors $G_n(\cdot) = G(\cdot)$ dans $\Omega \setminus \operatorname{supp}(\rho_n)$.*

Ce Théorème peut paraître surprenant car la suite (ρ_n) n'est évidemment pas monotone. Néanmoins, la condition (iii) de la définition signifie que les ρ_n ne doivent pas trop s'entrelacer. Le Théorème 0.2 se montre d'abord dans une boule en utilisant les coordonnées radiales puis est étendu grace à la linéarité du laplacien à des ouverts quelconques. Ce résultat reste valable pour des opérateurs linéaires elliptiques, ainsi que pour certains opérateurs quasi-linéaires comme le p-Laplacien (en restant dans la boule dans le cas non-linéaire).

Une autre question d'importance concerne l'approximation de la fonction de Green $\mathbb{G}(x, y)$. Nous montrons que si ρ_n est convenablement choisie, alors la suite de fonction $\mathbb{G}_n(\cdot,\cdot)$ définie par

$$\begin{cases} -\Delta_y \mathbb{G}_n(x, y) = \rho_n(x - y)\chi(x, y) & \text{dans } \Omega, \\ \mathbb{G}_n(x, y) = 0 & \text{sur } \partial\Omega, \end{cases} \tag{3}$$

converge vers la fonction de Green $\mathbb{G}(\cdot,\cdot)$ de manière monotone. Le Laplacien Δ_y désigne bien entendu le Laplacien classique par rapport à la seconde variable, y. Ici, $\chi(\cdot,\cdot)$ est la fonction continue dans Ω^2 définie par

$$\chi(x, y) = f\left(\frac{|y - x|}{r(x)}\right), \tag{4}$$

où $f \in C(\mathbb{R}_+)$ est décroissante, $f(r) = 1$ pour $0 \leqslant r \leqslant 1/2$, 0 si $r \geqslant 1$, et $r(x) = \operatorname{dist}(x, \partial\Omega)$. Alors on a le

Théorème 0.3. *Soit $\rho = (\rho_n)$ une bonne approximation de δ_0, radialement décroissante. Alors l'approximation $\mathbb{G}_n(\cdot,\cdot)$ obtenue par* (3) *est monotone et converge vers la fonction de Green $\mathbb{G}(\cdot,\cdot)$.*

Dire que l'approximation $\rho = (\rho_n)$ est radialement décroissante signifie pour tout $n \in \mathbb{N}$, la fonction $|x| \mapsto \rho_n(|x|)$ est décroissante. Dans ce théorème, il n'est pas nécessaire que le support de ρ_n soit inclus dans Ω, puisque ce sera le cas en multipliant par la fonction χ. Un corollaire intéressant en pratique est le suivant :

Corollaire 0.4. *Soit μ une mesure positive finie sur Ω et u la solution du problème*

$$\begin{cases} -\Delta u = \mu & \textit{dans } \Omega, \\ u = 0 & \textit{sur } \partial\Omega. \end{cases}$$

On note u_n la solution du même problème avec second-membre $\mu_n = (\rho_n\chi) \star \mu \in \mathrm{C}^\infty(\Omega)$. Alors, si $\rho = (\rho_n)$ satisfait les hypothèses du Théorème 0.3, *la suite (u_n) converge vers u de façon monotone dans Ω.*

1. Introduction

Let Ω be a regular bounded domain in $\mathbb{R}^d$, $d \geqslant 2$, containing the origin. We denote by $\mathbb{G}(x, y)$ the Green's function of the Laplacian (the reader will find a complete study of this subject in Bénilan [1]) in Ω, that is, the unique solution of the following problem:

$$\begin{cases} -\Delta\mathbb{G}(x,\cdot) = \delta_x(\cdot) & \text{in } \Omega, \\ \mathbb{G}(x,\cdot) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

We also denote $G(y) = \mathbb{G}(0, y)$ the solution obtained from a Dirac measure placed at $x = 0$, a point supposed to belong to Ω. A standard process to construct G is as follows: consider a resolution of the identity, that is, a sequence of smooth functions $\rho = (\rho_n)$, converging weakly to the Dirac measure, δ_0. Then it is well-known that for any $n \in \mathbb{N}$, the approximate problem:

$$\begin{cases} -\Delta G_n = \rho_n & \text{in } \Omega, \\ G_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

has a unique solution G_n and that the sequence (G_n) converges to the Green's function G. In this note, we prove that the convergence is monotone provided ρ satisfies some (natural) additional assumption. In all the note, $B(r) = B_r$ denotes the ball of radius $r > 0$ centered at the origin and $d \geqslant 2$ is the space dimension.

Definition 1.1. We say that the sequence $\rho = (\rho_n)$ is a good approximation of δ_0, the Dirac measure placed at $x = 0$ in Ω if it satisfies the following properties:

(i) $\rho_n \in \mathrm{C}^0(\mathbb{R}^d)$, $\rho_n \geqslant 0$, $\rho_n(x) = \rho_n(|x|)$ is radial, and $\mathrm{supp}(\rho_n) \subset \Omega$,
(ii) $\int_\Omega \rho_n(x)\,\mathrm{d}x = 1$, $\int_K \rho_n(x)\,\mathrm{d}x \to 0$ for any compact set $K \subset \Omega \setminus \{0\}$,
(iii) for any $n \in \mathbb{N}$, there exists a unique $\eta_n \in \mathbb{R}$ such that $B(\eta_n) \subset \Omega$ and

$$\begin{cases} \rho_n(x) < \rho_{n+1}(x) & \text{if } 0 \leqslant |x| < \eta_n, \\ \rho_n(x) > \rho_{n+1}(x) & \text{if } |x| > \eta_n. \end{cases} \tag{7}$$

Assumption (iii) can be understood as a one-intersection property. For instance, if ρ is radially decreasing, then $\rho_n(x) = n^d\rho(nx)$ has this property. We shall prove the

Theorem 1.2. *Let ρ be a good approximation of δ_0 in Ω. Then the following holds*:

(i) *The sequence (G_n) is monotone nondecreasing and converges to the function G in Ω.*
(ii) *If $\Omega \setminus \mathrm{supp}(\rho_n) \neq \emptyset$, then $G_n(x) = G(x)$ in $\Omega \setminus \mathrm{supp}(\rho_n)$.*

This result is somewhat surprising since the ρ_n will not enjoy any (global) comparison property. In fact, we will show that this property is shared by other operators, like $\mathcal{L}u = -\,\mathrm{div}(a(r)\nabla u)$ where $a(r)$ does not degenerate, and the p-Laplace operator, $-\Delta_p u = -\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

2. Monotone approximations of $G(0, y)$

We denote by r_n the radius of the support of ρ_n (recall that ρ_n is radial with support in Ω), so that $\mathrm{supp}(\rho_n) = B(r_n) \subset \Omega$. Theorem 1.2 is the combination of several steps. We first show the result in the case $\Omega = B(1)$, and then extend it for general domains. In the ball, all solutions are radial so that we use the notation $G_n(r)$ and $G_n'(r)$ for the radial derivative. We shall use several times the well-known Theorem of Gauss. In the context of Electrostatics, it says that the flux of the electric field E through a surface $\mathcal{S}$ equals the total electric charge inside $\mathcal{S}$. In the present situation, it may be seen as pure integration by parts (Green's formula):

$$\int_{\mathcal{V}} \rho_n(x)\,\mathrm{d}x = -\int_{\partial\mathcal{V}} \frac{\partial G_n}{\partial\nu}\,\mathrm{d}\sigma,$$

where $\mathcal{V}$ is any arbitrary closed regular volume of frontier $\partial\mathcal{V}$.

• STEP 1 – Let $\Omega = B_1$, then $G_n = G$ in $\omega_n = B_1 \setminus B(r_n)$ (if this set is non void).

Proof. We first use Gauss's theorem in the volume $\mathcal{V} = B_1$: since the solution is radial we get

$$1 = \int_{B_1} \rho_n(x)\,\mathrm{d}x = -\int_{\partial B_1} \frac{\partial G_n}{\partial\nu}\,\mathrm{d}\sigma = -|S|\cdot G_n'(1),$$

where $|S|$ stands for the measure of the unit sphere in $\mathbb{R}^d$. This shows that all the approximations have the same gradient at the boundary $|x| = 1$. The same also happens for the Green's function itself, G. Thus in the set ω_n, G and G_n satisfy the same equation with the same value (zero) at $|x| = 1$, and the same gradient at $|x| = 1$. Hence they are equal in ω_n. □

• STEP 2 – Comparison in the annulus: $G_n(x) \leqslant G_{n+1}(x)$ for $\eta_n \leqslant |x| \leqslant 1$.

Proof. We use again Gauss's theorem in the volume $\mathcal{V} = B_1 \setminus B_r$, for $r \in (\eta_n, 1)$. Subtracting the formulas for G_n and G_{n+1} yields:

$$\int_{B_1\setminus B_r} \big(\rho_{n+1}(x) - \rho_n(x)\big)\,\mathrm{d}x = |S|\big(G_{n+1}'(r) - G_n'(r)\big).$$

Indeed, the surface integral at $r = 1$ is null since $G_{n+1}'(1) = G_n'(1)$ (see Step 1). Hence, $G_{n+1}'(r) \leqslant G_n'(r)$ for $r \in (\eta_n, 1)$ because $\rho_{n+1} \leqslant \rho_n$ in this set. Since both solutions agree at $r = 1$, this implies that $G_{n+1}(r) \geqslant G_n(r)$ in $[\eta_n, 1]$. □

• STEP 3 – $G_n(x) \leqslant G_{n+1}(x)$ in $B(\eta_n)$.

Proof. It is just the maximum principle applied in $B(\eta_n)$. Indeed, $w = G_{n+1} - G_n$ satisfies $-\Delta w = \rho_{n+1} - \rho_n > 0$ in $B(\eta_n)$, thus w cannot have a minimum inside $B(\eta_n)$, so that the minimum is attained on the boundary. But from Step 2 we know that $w \geqslant 0$ on $|x| = \eta_n$, so that $w \geqslant 0$ everywhere in $B(\eta_n)$. This ends the proof of Theorem 1.2 in the case $\Omega = B_1$. □

• STEP 4 – Theorem 1.2 is still valid in Ω bounded and regular.

Proof. For simplicity we shall assume that $\Omega \subset B(1) = B$. Then we denote by G^Ω the Green's function in Ω and G^B the Green's function in the ball. A standard way to relate G^Ω to G^B is the following: if H is the unique solution of the regular problem:

$$\begin{cases} -\Delta H = 0 & \text{in } \Omega,\\ H(x) = G^B(x) & \text{on } \partial\Omega, \end{cases} \tag{8}$$

then it is clear that $G^{\Omega} = G^B - H$. The same method works also for the approximations: let H_n be the unique solution of

$$\begin{cases} -\Delta H_n = 0 & \text{in } \Omega, \\ H_n(x) = G_n^B(x) & \text{on } \partial\Omega. \end{cases}$$

Then it is immediate to check that $G_n^{\Omega} = G_n^B - H_n$. We proved in the previous Steps that G_n^B converges monotonically to G^B in the ball and that $G_n^B = G^B$ outside the support of ρ_n. But since $\operatorname{supp}(\rho_n) \subset \Omega$, then G_n^B agrees with G^B on $\partial\Omega$. It follows that $H_n \equiv H$, therefore

$$G_n^{\Omega} = G_n^B - H.$$

We deduce two properties from this information: firstly, for any $x \in \Omega \setminus \operatorname{supp}(\rho_n)$, $G_n^{\Omega}(x) = G^B(x) - H(x) = G^{\Omega}(x)$. Secondly, that the sequence G_n^{Ω} is monotone nondecreasing since the sequence G_n^B is, which ends the proof. □

Remark 1. It is clear that the same result holds if we only require that $\int_{\Omega} \rho_n(x)\,\mathrm{d}x$ increases to 1, instead of being constant to 1, except that in this case, the G_n never agree with the limit G in $\Omega \setminus \operatorname{supp}(\rho_n)$.

3. A monotone approximation of the Green's function $\mathbb{G}(x, y)$

In this section, we obtain a monotone approximation of the Green's function $\mathbb{G}(x, y)$, satisfying (5). A natural approximation of $\mathbb{G}$ is as follows: for fixed $x \in \Omega$,

$$\begin{cases} -\Delta_y \mathbb{G}_n(x, y) = \rho_n(x - y) & \text{for all } y \in \Omega, \\ \mathbb{G}_n(x, y) = 0 & \text{for all } y \in \partial\Omega. \end{cases} \tag{9}$$

Here, Δ_y denotes the Laplacian with respect to the second variable, y. However, if $x \in \Omega$ is fixed, it is not clear that $\rho_n(x - \cdot)$ is a good approximation of δ_x, the Dirac measure placed at x, in the sense that the support of $\rho_n(x - \cdot)$ does not necessarily lies in Ω. In fact, if x is close to the boundary, then for some values of n, $\operatorname{supp}(\rho_n(x - \cdot)) \setminus \Omega \neq \emptyset$. Thus, it is not clear that the approximation (9) yields a global monotone process $\{\mathbb{G}_n(\cdot, \cdot)\}$. More precisely, there may be a $n_0(x)$ after which the sequence $\mathbb{G}_n(x, \cdot)$ is monotone but we give below a suitable modification for which the process is indeed monotone, that is, an approximation for which $n_0(x) = 1$ for any $x \in \Omega$.

Let $f \in C(\mathbb{R}_+)$ decreasing, such that $f(r) = 1$ if $0 \leqslant r \leqslant 1/2$, and 0 if $r \geqslant 1$. Clearly, such function exists and if $r(x) = \operatorname{dist}(x, \partial\Omega)$, we put

$$\chi(x, y) = f\left(\frac{|x - y|}{r(x)}\right). \tag{10}$$

Then $\chi(\cdot, \cdot) \in C(\Omega^2)$ and for any fixed x, the function $y \mapsto \chi(x, y)$ is radially decreasing and its support lies within $B^x = B(x, r(x))$. Then we have:

Theorem 3.1. *Let $\rho = (\rho_n)$ be a good approximation of the delta placed at $x = 0$, radially decreasing (i.e., for any $n \in \mathbb{N}$, $|x| \mapsto \rho_n(|x|)$ is decreasing). Then the approximation $\mathbb{G}_n$ of $\mathbb{G}$ defined below is monotone: for $x \in \Omega$,*

$$\begin{cases} -\Delta_y \mathbb{G}_n(x, y) = \rho_n(x - y)\chi(x, y) & \textit{for all } y \in \Omega, \\ \mathbb{G}_n(x, y) = 0 & \textit{for all } y \in \partial\Omega. \end{cases} \tag{11}$$

Proof. Let us fix $x \in \Omega$ (arbitrary) and consider the nucleus $\rho_n^x(y) = \rho_n(x - y)\chi(x, y)$. Then the sequence ρ_n^x clearly enjoys properties: (i) and (ii) of Definition 1.1, except that the mass of ρ_n^x is not always one. In fact, if x is close to the boundary, the support of ρ_n may not lie entirely within B^x so that the mass calculated is strictly less than one. However, since ρ_n is radially decreasing, the mass of ρ_n^x increases with $n \geqslant 1$. On the other hand the number of intersections between ρ_n or ρ_{n+1} can be either zero or one. Note that if $\rho_n \leqslant \rho_{n+1}$ in B^x (zero intersection points) the comparison principle gives $\mathbb{G}_n(x,\cdot) < \mathbb{G}_{n+1}(x,\cdot)$.

Note also that if the support of ρ_n shrinks, then the mass is one and the one intersection property holds, for n sufficiently large. Then we use Theorem 1.2 and Remark 1 to conclude that the sequence $\{\mathbb{G}_n(x,\cdot)\}_{n\geqslant 1}$ converges monotonically to $\mathbb{G}(x,\cdot)$ in Ω. Since x is arbitrary, then we have proved that the whole sequence $\mathbb{G}_n(\cdot,\cdot)$ converges monotonically to the Green's function $\mathbb{G}$ in $\Omega \times \Omega$. □

Note that in the previous theorem, no restriction on the support of ρ_n is necessary since $\mathrm{supp}(\rho_n^x) \subset \Omega$, because of χ. This global approximation of $\mathbb{G}(x,y)$ has the following interesting application:

Corollary 3.2. *Let μ be a nonnegative finite measure on Ω and consider the problem*

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We denote by u_n the solution of the problem above with $\mu = (\rho_n\chi) \star \mu \in \mathrm{C}^\infty(\Omega)$. If ρ_n satisfies the same condition as in Theorem 3.1, *then the sequence (u_n) converges monotonically to u in Ω.*

The proof is clear: since $u_n = \mathbb{G} \star ((\rho_n\chi) \star \mu) = \mathbb{G}_n \star \mu$, and $\mu \geqslant 0$, the sequence is indeed monotone.

4. Comments

This method can be applied to a number of other situations: let us consider for instance some general elliptic operators in divergence form:

$$\mathcal{L}u := -\operatorname{div}\big(a(|x|)\,\nabla u\big). \tag{12}$$

If we assume that $a(\cdot)$ is continuous and that there exist two constants $m, M > 0$ such that $m \leqslant a(r) \leqslant M$, then the Green's function of the operator is well-defined, as the unique solution $\mathbb{G}$ of the problem: $\mathcal{L}u = \delta_0$ in Ω with zero boundary data. In this setting, Theorems 1.2 and 3.1 clearly extend to this type of operators.

Part of the method would apply also to the case $\mathcal{L}u := -\operatorname{div}(a(|x|, u, \nabla u)\nabla u)$, under some conditions on a. A typical example is the p-Laplace operator. But if Theorem 1.2 remains valid in a ball, we are unable to derive Theorem 3.1 due to the nonlinear character of the operator: we cannot extend to other set Ω than balls and we do not now how to deal with Dirac measures δ_x, placed at $x \neq 0$.

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