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Asymptotics of instability zones of Hill operators with a two term potential

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Abstract

We give a sharp asymptotics of the instability zones of the Hill operator $Ly = -y'' + (a\cos 2x + b\cos 4x)y$ for arbitrary real $a, b \neq 0$. ***To cite this article: P. Djakov, B. Mityagin, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Estimation asymptotique des intervalles d'instabilité d'opérateurs de Hill avec potentiels à deux termes. Dans cette Note on donne une estimation asymptotique des intervalles d'instabilité d'opérateurs de Hill de la forme $Ly = -y'' + (a\cos 2x + b\cos 4x)y$, où a et b sont des réels non nuls arbitraires. ***Pour citer cet article : P. Djakov, B. Mityagin, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1.

The Schrödinger operator $Ly = -y'' + v(x)y$, $-\infty < x < \infty$, with real valued periodic $L^2([0,\pi])$-potential $v(x)$, $v(x+\pi) = v(x)$, has spectral gaps, or instability zones $(\lambda_n^-, \lambda_n^+)$, $n \geqslant 1$, close to n^2 if n is large enough. The points λ_n^-, λ_n^+ could be determined as well as eigenvalues of the Hill equation $Ly = -y'' + v(x)y = \lambda y$, considered on $[0,\pi]$ with boundary conditions Per^+: $y(0) = y(\pi)$, $y'(0) = y'(\pi)$ for n even, and Per^-: $y(0) = -y(\pi)$, $y'(0) = -y'(\pi)$ for n odd. See details and basics in [15,17,18,21].

Let $\gamma_n = \lambda_n^+ - \lambda_n^-$ be the lengths of the spectral gaps. The decay rates of (γ_n) are in a close relation with smoothness of the potential v (see [11,12,22,4–6]). Sometimes the Lyapunov function, or the Hill discriminant (see [17], Sections 2.1–2.2) $\Delta(\lambda)$ can be written explicitly as it happens in the Krönig–Penney model, made of

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a periodic array of δ and δ' functions, or onionlike scatterers with several channels (see details in [2] and the bibliography there). Then the asymptotics of the roots of Lyapunov functions (trigonometric polynomials (7), (8) in [2]) and consequently the asymptotics of gaps and bands become a question about roots of elementary trigonometric functions. Without explicit Lyapunov function this task is much more difficult.

2.

In 1980 Harrell [10], and then Avron and Simon [1] gave the asymptotics of spectral gaps of the Mathieu operator $-\frac{d^2}{dx^2} + 2a\cos 2x$. They showed that

$$\gamma_n = \lambda_n^+ - \lambda_n^- = 8\left(\frac{|a|}{4}\right)^n \frac{1}{((n-1)!)^2}\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

In [1] the question was raised about these asymptotics in the case of two term potential

$$v(x) = a\cos 2x + b\cos 4x. \tag{1}$$

Later, Grigis [9] gave generic asymptotics of spectral gaps of the Schrödinger operator $-\frac{d^2}{dx^2} + v(x)$ when v is a real-valued trigonometric polynomial. For him, the two term potential

$$u(x) = c\sin 2x + d\cos 4x, \quad d > 0, \tag{2}$$

was of special interest as well. (Notice that the shift $x \to x + \pi/4$ transforms $u(x) \in (2)$ into $v \in (1)$ with $a = c$, $b = -d$. Their Schrödinger operators are isospectral, so we can consider without loss of generality just potentials (1); however, b could be positive or negative.)

3.

Recently, we found [7,8] the asymptotics of (γ_n) for a potential of the form (2) when $c^2 = 8d > 0$. Our proofs were based on the relationship of Dirac operator with potential $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$ and Hill operators with potential $u = \pm p' + p^2$, the Ricatti transform of p. In terms of a, b in (1), if we introduce a parameter t by

$$a^2 + 8bt^2 = 0, \tag{3}$$

then $c^2 = 8d > 0$ is a special case of (3) with $t = \pm 1$. Generally, for real $a, b \neq 0$ we set $a^2 + 8bt^2 = 0$, $b = -2\alpha^2$, $a = -4\alpha t$, where

(i) α and t are real if $b < 0$,
(ii) α and t are pure imaginary if $b > 0$.

Now, this parametrization plays a special role in asymptotic behavior of gaps $\gamma_n(\alpha)$, both for $\alpha \to 0$ and $n \to \infty$.

Theorem 3.1. *Let γ_n, $n \in \mathbb{N}$, be the spectral gaps (lengths of instability zones) of the operator*

$$Ly = -y'' - [4\alpha t\cos 2x + 2\alpha^2\cos 4x]y. \tag{4}$$

If t and n are fixed, then for even n

$$\gamma_n = \frac{\pm 8\alpha^n}{2^n[(n-1)!]^2}\prod_{k=1}^{n/2}\left(t^2 - (2k-1)^2\right)\left(1 + O(\alpha)\right), \tag{5}$$

and for odd n

$$\gamma_n = \frac{\pm 8\alpha^n t}{2^n[(n-1)!]^2} \prod_{k=1}^{(n-1)/2} \bigl(t^2 - (2k)^2\bigr)\bigl(1 + \mathrm{O}(\alpha)\bigr). \tag{6}$$

Remark 1. In the case (ii), if we put $\alpha = \mathrm{i}\beta$, $t = \mathrm{i}\tau$, β, τ real, then we can rewrite, say, (6), as

$$\gamma_n = \frac{\pm 8\beta^n \tau}{2^n[(n-1)!]^2} \prod_{k=1}^{(n-1)/2} \bigl(\tau^2 + (2k)^2\bigr)\bigl(1 + \mathrm{O}(\beta)\bigr).$$

Of course, (5) could be rewritten in terms of β, τ in the same way.

Proof is based, on the one hand, on our analytic methods [3–6], and on the other hand, on using the approach to coexistence problem of Magnus and Winkler (see [16], [17], Chapter 7, in particular, Theorem 7.9) and sharpening their result about the multiplicities of eigenvalues of the operator (4) in the case where t is an integer.

4.

The essential components of the asymptotics (5) and (6) are polynomials in t of degree n. The combinatorial meaning of their coefficients unearthed in the course of the proof of Theorem 3.1 leads to a series of algebraic identities.

Theorem 4.1. *The following formulae hold*:

$$\sum (m^2 - i_1^2) \cdots (m^2 - i_k^2) = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant m} (2j_1 - 1)^2 \cdots (2j_k - 1)^2, \tag{7}$$

where the first sum is over all indicies i_s such that

$$-m < i_1 < \cdots < i_k < m, \quad |i_s - i_r| \geqslant 2;$$

$$\sum \bigl[(2m-1)^2 - (2i_1 - 1)^2\bigr] \cdots \bigl[(2m-1)^2 - (2i_k - 1)^2\bigr] = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant m-1} (4j_1)^2 \cdots (4j_k)^2, \tag{8}$$

where the first sum is over all indicies i_s such that

$$-m + 1 < i_1 < \cdots < i_k < m, \quad |i_s - i_r| \geqslant 2.$$

The terms in (7) and (8) look to be similar to the terms in the identity conjectured by Kac and Wakimoto [13] and proved by Milne [19], and later by Zagier [23]; see details and further bibliography in [20], in particular, Section 7 and Corollary 7.6, pp. 120–121. Our asymptotic analysis involves eigenvalues of Schrödinger operators. This occurrence of eigenvalues suggests a possible link with advanced determinant calculus developed by Andrews (see Krattenthaler [14] and references there) and Hankel determinants in Milne [20].

5. Asymptotics for $n \to \infty$

Theorem 5.1. *Under the notations of Theorem* 3.1, *let $\alpha \neq 0$ and $t \neq 0$ be fixed. Then for even n*

$$\gamma_n = \pm \frac{8\alpha^n}{2^n[(n-2)!!]^2} \cos\left(\frac{\pi}{2} t\right)\left[1 + \mathrm{O}\left(\frac{\log n}{n}\right)\right], \tag{9}$$

and for odd n

$$\gamma_n = \pm \frac{8\alpha^n}{2^n[(n-2)!!]^2}\frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\left[1+\mathrm{O}\left(\frac{\log n}{n}\right)\right]. \tag{10}$$

[Let us recall that $(2k-1)!! = 1\cdot 3\cdots(2k-1)$, $(2k)!! = 2\cdot 4\cdots 2k$.]

Remark 2. As in Remark 1, in the case (ii), $\cos(\pi t/2) = \cosh(\pi\tau/2)$ and if n is odd, then the product $\alpha^n \sin(\pi t/2) = \mathrm{i}^{n+1}\beta^n \sinh(\pi\tau/2)$ is real.

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