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Complex Analysis

A regularisation of the Coleff–Herrera residue current

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Abstract

We prove that if a holomorphic mapping from some complex manifold to $\mathbb{C}^2$ defines a complete intersection then the corresponding Coleff–Herrera residue current can be smoothly regularised by a $(0, 2)$-form depending on two parameters. ***To cite this article: H. Samuelsson, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Une régularisation du courant résiduel de Coleff–Herrera. Nous démontrons que, si une application holomorphe d'une variété complexe à valeurs dans $\mathbb{C}^2$ définit une intersection complète, alors le courant résiduel de Coleff–Herrera correspondant peut être régularisé par une $(0, 2)$-forme dépendant de deux paramètres. ***Pour citer cet article : H. Samuelsson, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Soit X une variété complexe et soit $f = (f_1, f_2) : X \to \mathbb{C}^2$ une application holomorphe. Supposons que f définit une intersection complète, autrement dit que la variété $V_f = \{f_1 = f_2 = 0\}$ est de codimension 2. Définissons, pour toute forme $\varphi \in \mathcal{D}_{n,n-2}(X)$, l'intégrale résiduelle

$$I_f^{\varphi}(\epsilon_1, \epsilon_2) := \int_{\substack{|f_1|^2=\epsilon_1 \\ |f_2|^2=\epsilon_2}} \frac{\varphi}{f_1 f_2}.$$

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Le courant résiduel de Coleff–Herrera, noté $[\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}]$, est défini comme la limite de cette intégrale lorsque ϵ_1 et ϵ_2 tendent vers zéro le long d'un « chemin admissible », ce qui, dans ce cadre, signifie que, par exemple, ϵ_1 tend vers zéro plus vite que toute puissance de ϵ_2 (cf. [3]). Il est bien connu que l'integrale résiduelle est, en général, dicontinue à l'origine [7,2]. Dans cette note, nous donnons une démonstration du résultat suivant.

Theorem 0.1. *Soit X une variété complexe et soit $f=(f_1,f_2):X\to\mathbb{C}^2$ une application holomorphe. Supposons que f définit une intersection complète. Alors*

$$\lim_{\epsilon_1,\epsilon_2\to 0}\int\bar\partial\frac{\bar f_1}{|f_1|^2+\epsilon_1}\wedge\bar\partial\frac{\bar f_2}{|f_2|^2+\epsilon_2}\wedge\varphi=\left[\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}\right].\varphi$$

pour toute forme $\varphi\in\mathcal{D}_{n,n-2}(X)$.

Par conséquent, le courant de Coleff–Herrera peut être obtenu comme la limite (au sens des courants) d'une $(0,2)$-forme régulière dépendant de deux paramètres, indépendamment de la façon dont on s'approche de l'origine.

1. Introduction and the result

Let X be a complex manifold and let $f=(f_1,f_2):X\to\mathbb{C}^2$ be a holomorphic mapping. Assume that f defines a complete intersection, i.e. that $V_f=\{f_1=f_2=0\}$ has codimension 2 in X. The corresponding *Coleff–Herrera residue current* was originally defined as follows [3]. Denote the *residue integral* by

$$I_f^{\varphi}(\epsilon_1,\epsilon_2):=\int\limits_{\substack{|f_1|^2=\epsilon_1\\|f_2|^2=\epsilon_2}}\frac{\varphi}{f_1f_2},$$

where φ is any test form of bidegree $(n,n-2)$. If we let ϵ_1 and ϵ_2 approach the origin along an 'admissible path', which in this context means that ϵ_1 tends to zero faster then any power of ϵ_2 or vice versa, then the residue integral has a limit independently of the choice of admissible path and this limit defines the action of a $(0,2)$-current, the Coleff–Herrera residue current, on the test form φ. We will denote this current by $[\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}]$. It is known that an unrestricted limit of the residue integral does not exist in general. Passare and Tsikh showed in [7] that if we take $f_1=z_1^4$, $f_2=z_1^2+z_2^2+z_1^3$ and a test form which in a neighbourhood of the origin equals $\varphi(z)=\bar z_2f_2(z)\,dz_1\wedge dz_2$ then the residue integral has limit zero if we approach the origin along any path $\delta\mapsto(\delta^4,c\delta^2)$, $c\neq 1$ and a nonzero limit if we approach the origin along the path $\delta\mapsto(\delta^4,\delta^2)$. Other examples disproving the continuity of the residue integral at the origin have been found by Björk [2]. The aim of this note is to outline a proof of the following result saying that the Coleff–Herrera current can be obtained as the unrestricted (weak) limit of a smooth $(0,2)$-form depending on two parameters.

Theorem 1.1. *Let X be a complex manifold and let $f=(f_1,f_2):X\to\mathbb{C}^2$ be a holomorphic mapping. Assume that f defines a complete intersection in X. Then*

$$\lim_{\epsilon_1,\epsilon_2\to 0}\int\bar\partial\frac{\bar f_1}{|f_1|^2+\epsilon_1}\wedge\bar\partial\frac{\bar f_2}{|f_2|^2+\epsilon_2}\wedge\varphi=\left[\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}\right].\varphi$$

for all test forms φ of bidegree $(n,n-2)$.

Before we continue with the proof section we mention that a thorough study of the limits of the residue integral along paths of the form $\delta\mapsto(\delta^{s_1},\delta^{s_2})$ for $(s_1,s_2)\in\mathbb{R}_+$ has been done by Passare in [5]. He shows that as long as (s_1,s_2) avoids finitely many lines through the origin the corresponding limit of the residue integral equals the limit along an admissible path. We also mention, and will later use, an alternative approach to the Coleff–Herrera residue current proposed by Passare and Tsikh [6]. If we compute the (iterated) Mellin transform of the residue integral we get

$$\int \frac{\bar\partial |f_1|^{2\lambda_1}}{f_1} \wedge \frac{\bar\partial |f_2|^{2\lambda_2}}{f_2} \wedge \varphi$$

at least for the real part of λ_1 and λ_2 large enough. Passare and Tsikh showed that the integral as a function of λ_1 and λ_2 has an analytic continuation to a neighbourhood of the origin in $\mathbb{C}^2$ and that the value at $\lambda_1 = \lambda_2 = 0$ equals the limit of the residue integral along an admissible path.

2. Outline of the proof

We first present and indicate how to prove two technical results, Propositions 2.1 and 2.2, and then we finish the proof of Theorem 1.1 using these results.

Proposition 2.1. *Let Ψ and Φ be strictly positive smooth functions on $\mathbb{C}^n$. Then for any $\varphi \in \mathcal{D}_{n,n}(\mathbb{C}^n)$ we have*

$$\lim_{\epsilon_1,\epsilon_2\to 0^+} \int \frac{\bar\zeta^\alpha \Psi}{|\zeta^\alpha|^2\Psi + \epsilon_1} \frac{\bar\zeta^\beta \Phi}{|\zeta^\beta|^2\Phi + \epsilon_2}\varphi = \left[\frac{1}{\zeta^{\alpha+\beta}}\right].\varphi.$$

Proposition 2.2. *Let Ψ and Φ be strictly positive smooth functions on $\mathbb{C}^n$. Then for any $\varphi \in \mathcal{D}_{n,n}(\mathbb{C}^n)$ we have*

$$\lim_{\epsilon_1,\epsilon_2\to 0^+} \epsilon_2 \int \frac{\bar\zeta^{\alpha+\beta}}{(|\zeta^\alpha|^2\Psi + \epsilon_1)(|\zeta^\beta|^2\Phi + \epsilon_2)^2}\varphi = 0.$$

The key to understanding Propositions 2.1 and 2.2 is the next lemma which maybe also has some independent interest. It is a version of Taylor's formula but unlike the usual one that gives us a polynomial approximation in a neighbourhood of the intersection of the coordinate hyperplanes our version provides us with an approximation in a neighbourhood of the union of the coordinate hyperplanes. Our approximation is, in general, not a polynomial, however, but has enough similarities for our purposes. Define the linear operator $M_j^{r_j}$ on $C^\infty(\mathbb{C}^n)$ to be the operator that maps φ to the Taylor polynomial of degree r_j of the function $\zeta_j \mapsto \varphi(\zeta)$ (centered at $\zeta_j = 0$). A straight forward computation shows that $M_j^{r_j}$ and $M_i^{r_i}$ commute.

Lemma 2.3. *Let $K \subseteq \{1, \dots, n\}$ and $r = (r_{i_1}, \dots, r_{i_{|K|}})$ and define the linear operator M_K^r on $C^\infty(\mathbb{C}^n)$ by*

$$M_K^r = \sum_{j\in K} M_j^{r_j} - \sum_{\substack{i,j\in K\\ i<j}} M_i^{r_i} M_j^{r_j} + \cdots + (-1)^{|K|+1} M_{i_1}^{r_{i_1}} \cdots M_{i_{|K|}}^{r_{i_{|K|}}}.$$

Then for any $\varphi \in C^\infty(\mathbb{C}^n)$ we have

$$\varphi - M_K^r\varphi = \mathcal{O}\left(\prod_{i\in K} |\zeta_i|^{r_i+1}\right).$$

Moreover, $M_K^r\varphi$ can be written as a (finite) sum of terms $\phi_{I,J}(\zeta)\zeta^I\bar\zeta^J$ where $I_i + J_i \leqslant r_i$ for $i \in K$ and $\phi_{I,J}(\zeta)$ is independent of the coordinate ζ_i if $I_i + J_i > 0$, and also if L is the set of indices $i \in K$ such that $I_i + J_i = 0$ then $\phi_{I,J}(\zeta) = \mathcal{O}(\prod_{i\in L} |\zeta_i|^{r_i+1})$.

It is now quite easy to see that Propositions 2.1 and 2.2 hold in the case Ψ and Φ are constant (for simplicity equal to 1). We illustrate by considering Proposition 2.2. Choose $K = \{1, \dots, n\}$ and $r = \alpha + \beta - 2$ and add and subtract $M_K^r \varphi$. Then the integral in Proposition 2.2 splits into

$$\epsilon_2 \int\limits_{\Delta} \frac{\bar{\zeta}^{\alpha+\beta}}{(|\zeta^\alpha|^2 + \epsilon_1)(|\zeta^\beta|^2 + \epsilon_2)^2} M_K^r \varphi \tag{1}$$

$$+ \epsilon_2 \int\limits_{\Delta} \frac{\bar{\zeta}^{\alpha+\beta}}{(|\zeta^\alpha|^2 + \epsilon_1)(|\zeta^\beta|^2 + \epsilon_2)^2} (\varphi - M_K^r \varphi), \tag{2}$$

where Δ is a big polydisc containing the support of φ. The integral in Eq. (1) is zero for all positive ϵ_1 and ϵ_2 by anti-symmetry since the terms in $M_K^r \varphi$ are polynomials in at least one of the variables. On the other hand, the integrand in Eq. (2) is locally integrable when $\epsilon_1 = \epsilon_2 = 0$, and so by the Dominated Convergence Theorem the limit of Eq. (2) equals the integral of the pointwise limit of the integrand and this is zero (almost everywhere). In the general case when Ψ and Φ are not constant we cannot use anti-symmetry directly to see that certain integrals vanishes. However we can use the following two results to see that it actually is enough anti-symmetry left in the general case to deduce the same thing. With the notation from Lemma 2.3 we have

$$\frac{\Psi}{\Psi + a/b} \frac{\Phi}{\Phi + c/d} = M_K^r \left(\frac{\Psi}{\Psi + a/b} \frac{\Phi}{\Phi + c/d} \right) + \prod_{i \in K} \left| \zeta_i^{r_i+1} \right| F(a, b, c, d, \zeta), \tag{3}$$

$$\frac{c/d}{(\Psi + a/b)(\Phi + c/d)^2} = M_K^r \left(\frac{c/d}{(\Psi + a/b)(\Phi + c/d)^2} \right) + \prod_{i \in K} \left| \zeta_i^{r_i+1} \right| \tilde{F}(a, b, c, d, \zeta), \tag{4}$$

where F and $\widetilde{F}$ are bounded on $(0, \infty)^4 \times D$ if $D \Subset \mathbb{C}^n$. The homogeneity in Eqs. (3) and (4) enables us to re-write the integrals in Propositions 2.1 and 2.2 in such a way that we can use anti-symmetry but we skip the details. We can now finish the proof of Theorem 1.1 but first we need some terminology for multiindices. We say that two multiindices α and β with the same number of components are *disjoint* if $\alpha_i \neq 0$ implies that $\beta_i = 0$ and $\beta_i \neq 0$ implies that $\alpha_i = 0$.

Proof. [Proof of Theorem 1.1.] We prove the following slightly stronger statement

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \int \frac{\bar{f}_1}{|f_1|^2 + \epsilon_1} \bar{\partial} \frac{\bar{f}_2}{|f_2|^2 + \epsilon_2} \wedge \varphi = \left[\frac{1}{f_1} \bar{\partial} \frac{1}{f_2} \right] . \varphi, \tag{5}$$

where φ is any test form of bidegree $(n, n-1)$. We will use the analytic continuation definition of the right-hand side of Eq. (5) [6,1], that is we will use that

$$\left[\frac{1}{f_1} \bar{\partial} \frac{1}{f_2} \right] . \varphi = \int \frac{|f_1|^{2\lambda}}{f_1} \frac{\bar{\partial} |f_2|^{2\lambda}}{f_2} \wedge \varphi \bigg|_{\lambda=0}.$$

By Hironaka's theorem [4], for any sufficiently small open $U \subset X$ we can find a complex manifold $\widetilde{U}$ and a proper holomorphic map $\pi : \widetilde{U} \to U$ which is a biholomorphism outside the null-set $\pi^* \{f_1 \cdot f_2 = 0\}$ such that $\{\pi^* f_1 \cdot \pi^* f_2 = 0\}$ has normal crossings in $\widetilde{U}$. Hence locally in $\widetilde{U}$ we can choose coordinates such that $\pi^* f_1 = \mu_1 g_1$ and $\pi^* f_2 = \mu_2 g_2$ where μ_j are monomials and g_j are non-vanishing holomorphic functions. After a partition of unity we may assume that φ has support in such a U, and so we see that in order to prove Eq. (5) it suffices to prove

$$\int \frac{\bar{\mu}_1 \bar{g}_1}{|\mu_1 g_1|^2 + \epsilon_1} \bar{\partial} \frac{\bar{\mu}_2 \bar{g}_2}{|\mu_2 g_2|^2 + \epsilon_2} \wedge \rho \pi^* \varphi \to \int \frac{|\mu_1 g_1|^{2\lambda}}{\mu_1 g_1} \frac{\bar{\partial} |\mu_2 g_2|^{2\lambda}}{\mu_2 g_2} \wedge \rho \pi^* \varphi \bigg|_{\lambda=0}, \tag{6}$$

where ρ is a cut-off function in $\widetilde{U}$. We write the monomials μ_j in local coordinates ζ on $\widetilde{U}$ as $\mu_1 = \zeta^\alpha \zeta^\gamma$ and $\mu_2 = \zeta^\beta \zeta^\delta$ where the multiindices α, β and γ are pairwise disjoint and $\gamma_j = 0$ if and only if $\delta_j = 0$. Hence α, β

and δ are also pairwise disjoint. Note that coordinate ζ_j divides both μ_1 and μ_2 if and only if $\gamma_j \neq 0$ or equivalently $\delta_j \neq 0$. The right-hand side of Eq. (6) can be computed by integrations by parts as in e.g. [1] and the result can be written

$$\left[\frac{1}{\zeta^{\alpha+\gamma+\delta}}\right] \otimes \bar\partial\left[\frac{1}{\zeta^{\beta}}\right].\frac{\rho\pi^*\varphi}{g_1g_2}. \tag{7}$$

Let K and L be the set of indices j such that $\beta_j \neq 0$ and $\gamma_j \neq 0$ respectively. Decompose the $\bar\partial$-operator as $\bar\partial = \bar\partial_K + \bar\partial_{K^c}$ where $\bar\partial_K$ and $\bar\partial_{K^c}$ are the parts corresponding to the variables ζ_j with $j \in K$ and $j \notin K$ respectively. Integrating by parts we see that the integral on the right-hand side of Eq. (6) equals

$$-\int \bar\partial_K\left(\frac{\bar\zeta^{\alpha+\gamma}\bar g_1}{|\zeta^{\alpha+\gamma}|^2|g_1|^2+\epsilon_1}\rho\pi^*\varphi\right)\frac{\bar\zeta^{\beta+\delta}\bar g_2}{|\zeta^{\beta+\delta}|^2|g_2|^2+\epsilon_2} \tag{8}$$

$$+\,\epsilon_2\int \frac{\bar\zeta^{\alpha+\gamma}\bar g_1}{|\zeta^{\alpha+\gamma}|^2|g_1|^2+\epsilon_1}\frac{\bar\zeta^{\beta}\bar\partial_{K^c}(\bar\zeta^{\delta}\bar g_2)}{(|\zeta^{\beta+\delta}|^2|g_2|^2+\epsilon_2)^2}\wedge\rho\pi^*\varphi. \tag{9}$$

Let us first consider Eq. (8). When $\bar\partial_K$ falls on $\rho\pi^*\varphi$ we get an integral which can be handled by Proposition 2.1 and in the limit we get $-[1/\zeta^{\alpha+\beta+\gamma+\delta}].\bar\partial_K\frac{\rho\pi^*\varphi}{g_1g_2}$ which is precisely Eq. (7). On the other hand when $\bar\partial_K$ falls on the quotient we get, since β is disjoint with both α and γ, an integral which by Proposition 2.2 tends to zero. It remains to see that also Eq. (9) tends to zero. When $\bar\partial_{K^c}$ falls on $\bar g_2$ we run into no problems and Proposition 2.2 says that this integral tends to zero. It is a bit more delicate when $\bar\partial_{K^c}$ falls on $\bar\zeta^{\delta}$ because then we get

$$\sum_{i\in L}\epsilon_2\delta_i\int\frac{\bar\zeta^{\alpha+\gamma}}{|\zeta^{\alpha+\gamma}|^2|g_1|^2+\epsilon_1}\frac{\bar\zeta^{\beta+\delta}}{(|\zeta^{\beta+\delta}|^2|g_2|^2+\epsilon_2)^2}\bar g_1\bar g_2\frac{\mathrm{d}\bar\zeta_i}{\bar\zeta_i}\wedge\rho\pi^*\varphi.$$

Now for the first and only time we have to use that V_f has codimension 2. We use the Coleff–Herrera trick to see that $\pi^*\varphi$ is a sum of terms which are either divisible by $\bar\zeta_j$ or $\mathrm{d}\bar\zeta_j$. In fact, if we let z be local coordinates on our original manifold X, then we can write

$$\varphi = \sum_{|I|=n-1}\varphi_I\wedge\mathrm{d}\bar z^I,$$

where the φ_I are $(n,0)$-forms. Since V_f has codimension 2, the $(0,n-1)$-forms $\mathrm{d}\bar z^I$ vanish on V_f. Hence $\pi^*\,\mathrm{d}\bar z^I$ vanishes on π^*V_f and in particular, since ζ_j divides both μ_1 and μ_2 for $j\in L$, it vanishes on $\{\zeta_j=0\}$. Moreover, $\partial\pi^*\,\mathrm{d}\bar z^I = \pi^*\partial\,\mathrm{d}\bar z^I = 0$, and so if we write

$$\pi^*\,\mathrm{d}\bar z^I = \sum_{|J|=n-1}C_J(\zeta)\,\mathrm{d}\bar\zeta^J,$$

we see that the coefficients $C_J(\zeta)$ must be anti-holomorphic. Hence if $\mathrm{d}\bar\zeta_i$ does not divide $\mathrm{d}\bar\zeta^J$ then $\bar\zeta_j$ must divide $C_J(\zeta)$ since $C_J(\zeta)$ is anti-holomorphic and zero on $\{\zeta_j=0\}$. Thus for $j\in L$ the form $\frac{\mathrm{d}\bar\zeta_j}{\bar\zeta_j}\wedge\pi^*\varphi$ is actually smooth (and compactly supported) and so we can use Proposition 2.2 to see that all the integrals in the sum above tend to zero. □

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