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Numerical Analysis

Stability of the finite element Stokes projection in $W^{1,\infty}$

Vivette Girault [a], Ricardo H. Nochetto [b], Ridgway Scott [c]

[a] *Laboratoire Jacques-Louis Lions, université P. et M. Curie, 75252 Paris cedex 05, France*
[b] *Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742–4015, USA*
[c] *Department of Mathematics and the Computation Institute, University of Chicago, Chicago, IL 60637–1581, USA*

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Abstract

We prove stability of the finite element Stokes projection in the product space $W^{1,\infty}(\Omega) \times L^{\infty}(\Omega)$. The proof relies on weighted L^2 estimates for regularized Green's functions associated with the Stokes problem and on a weighted inf–sup condition. The domain is a polygon or a polyhedron with a Lipschitz-continuous boundary, satisfying suitable sufficient conditions on the inner angles of its boundary, so that the exact solution is bounded in $W^{1,\infty}(\Omega) \times L^{\infty}(\Omega)$. The family of triangulations is shape-regular and quasi-uniform. The finite element spaces satisfy a super-approximation property, which is shown to be valid for commonly used stable finite element spaces. ***To cite this article: V. Girault et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Stabilité dans $W^{1,\infty}$ de la projection de Stokes par éléments finis. Nous démontrons que la norme du maximum du gradient de la vitesse et celle de la pression, calculés par des méthodes d'éléments finis usuelles pour discrétiser le problème de Stokes, sont bornées indépendamment du pas de la discrétisation. La démonstration est basée sur des estimations à poids dans L^2 pour des fonctions de Green associées au problème de Stokes et sur une condition inf–sup à poids. Le domaine est un polygone ou un polyèdre à frontière lipschitzienne dont les angles intérieurs satisfont des conditions suffisantes convenables pour assurer que la solution exacte est aussi bornée dans $W^{1,\infty}(\Omega) \times L^{\infty}(\Omega)$. La famille de triangulations est uniformément régulière. Nous employons une propriété de super-approximation que nous démontrons pour des espaces d'éléments finis couramment utilisés. ***Pour citer cet article : V. Girault et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Le but de cette Note est de montrer que le gradient de la vitesse de la projection de Stokes discrétisée par une méthode d'éléments finis usuelle, ainsi que sa pression discrète associée, sont bornés dans la norme du maximum. Le problème est posé dans un domaine Ω polygonal ou polyèdrique à frontière lipschitzienne, en dimension $d = 2$

E-mail addresses: girault@ann.jussieu.fr (V. Girault), rhn@math.umd.edu (R.H. Nochetto), ridg@cs.uchicago.edu (R. Scott).

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ou 3, décomposé par une triangulation $\mathcal{T}_h$ formée de d-simplexes, h étant le pas de la discrétisation. On se donne une vitesse $\boldsymbol{u} \in H_0^1(\Omega)^d$, à divergence nulle, une pression $p \in L_0^2(\Omega)$, i.e. à moyenne nulle, une triangulation $\mathcal{T}_h$ de $\overline{\Omega}$, et deux espaces d'éléments finis $X_h \subset H_0^1(\Omega)^d$ and $M_h \subset L_0^2(\Omega)$, ayant des propriétés convenables d'approximation et satisfaisant une condition inf–sup discrète uniforme. On définit $\boldsymbol{u}_h \in X_h$ et $p_h \in M_h$ par :

$$\int_\Omega \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \int_\Omega p_h \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} \quad \forall \boldsymbol{v}_h \in X_h, \tag{1}$$

$$\int_\Omega q_h \operatorname{div} \boldsymbol{u}_h \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall q_h \in M_h. \tag{2}$$

Notre résultat principal est : sous des hypothèses adéquates sur les angles de Ω et sur $\mathcal{T}_h$, si $\boldsymbol{u}$ appartient à $W^{1,\infty}(\Omega)^d$ et p à $L^\infty(\Omega)$, alors il existe une constante C indépendante de h, $\boldsymbol{u}$ et p, telle que

$$\|\nabla \boldsymbol{u}_h\|_{L^\infty(\Omega)} + \|p_h\|_{L^\infty(\Omega)} \leqslant C\big(\|\nabla \boldsymbol{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)}\big). \tag{3}$$

Ce résultat important est utile chaque fois qu'on discrétise un problème d'écoulement dont l'analyse demande une borne $W^{1,\infty}$ sur la vitesse. C'est le cas par exemple de certains écoulements à frontière libre ou des écoulements de fluides non-Newtoniens, hautement non-linéaires. Mais sa démonstration est difficile parce que (1) ne se prête pas à la norme de $W^{1,\infty}$.

Un résultat analogue pour le Laplacien a été démontré en plusieurs étapes, voir l'historique dans Schatz [11]. La première approche était une application du principe du maximum (cf. Ciarlet et Raviart [3]), mais la démonstration n'a vraiment été optimale qu'en 1982 (cf. Rannacher et Scott [10]) pour la dimension deux et en 1994 (cf. Brenner et Scott [1]) pour la dimension trois. Elle est basée sur des estimations à poids dans L^2 avec le poids :

$$\sigma(\boldsymbol{x}) = \big(|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (\kappa h)^2\big)^{1/2}, \quad \kappa > 1, \tag{4}$$

pris avec l'exposant

$$\mu = d + \lambda, \quad 0 < \lambda < 1, \tag{5}$$

où $\boldsymbol{x}_0$ est un point proche de celui où le maximum est atteint et λ et κ, independants de h, sont deux paramètres à choisir. Malheureusement, cette démonstration pour le Laplacien ne s'étend pas au problème de Stokes, parce qu'elle ne permet pas l'élimination de la pression. En effet, on trouve l'estimation habituelle :

$$\|\nabla \boldsymbol{u}_h\|_{L^2(\Omega)} \leqslant \|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}, \tag{6}$$

en prenant dans (1) des fonctions test qui vérifient (2), donc qui éliminent la pression. Mais dès qu'on introduit un poids, celui-ci multiplie la fonction test et le produit perd la propriété (2). Or la présence de la pression semble introduire un facteur logarithme en h, donc qui ne permet pas de démontrer la stabilité. Pour cette raison, le seul résultat publié du même type, celui de Durán et al. [6] en 1988, n'est pas optimal car dans ce travail la constante C de (3) a le facteur $|\log h|^{1/2}$. Une autre approche est présentée par Chen [2] dans une prépublication. Ce travail n'a pas de facteur logarithmique, mais il est restreint à un domaine de frontière très régulière.

Or en 2001, Durán et Muschietti [5] on démontré une condition inf–sup uniforme, dans un domaine Lipschitz, avec le poids σ^α pour tout exposant α tel que $-d/2 < \alpha < d/2$, et ont exhibé le facteur $|\log h|$ dans le cas critique où $|\alpha| = d/2$: pour tout $f \in L_0^2(\Omega)$, il existe $\boldsymbol{v} \in H_0^1(\Omega)^d$ tel que

$$\operatorname{div} \boldsymbol{v} = f \quad \text{dans } \Omega,$$
$$\big\|\sigma^{\alpha/2} \nabla \boldsymbol{v}\big\|_{L^2(\Omega)} \leqslant C_\alpha \big\|\sigma^{\alpha/2} f\big\|_{L^2(\Omega)}, \tag{7}$$

où la constante C_α ne dépend pas de h, κ, f ou $\boldsymbol{v}$. Leur preuve se fonde sur un théorème de Stein [12] qui démontre une borne précise sur une integrale singulière avec le poids $|\boldsymbol{x}|^\alpha$ pour $-d/2 < \alpha < d/2$.

Dans cette Note, nous adaptons l'analyse de [1] au problème de Stokes, nous transformons la contribution de la pression discrète de telle sorte que la condition inf–sup soit seulement utilisée dans des cas non critiques et ainsi nous éliminons le facteur logarithmique.

1. Main steps of the proof for the velocity

We present briefly the main points in the proof; the details are written in Girault et al. [7].

Reduction to weighted estimates. This step is standard and consists in reducing the estimate for $\boldsymbol{u}_h$ in $W^{1,\infty}$ into an error estimate for a regularized Green's function first in $W^{1,1}$, and next in H^1 with a weight. For this, we fix an element of the matrix $\nabla\boldsymbol{u}_h$, say $\partial\boldsymbol{u}_{h,i}/\partial x_j$, we choose a suitable point $\boldsymbol{x}_0$ located in the element T (triangle or tetrahedron) where $|\partial\boldsymbol{u}_{h,i}/\partial x_j|$ is maximum, and an approximate mollifier δ_M supported by T, satisfying:

$$\int_\Omega \delta_M \,\mathrm{d}\boldsymbol{x} = 1, \qquad \left\|\frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}\right\|_{L^\infty(\Omega)} = \left|\int_\Omega \delta_M \frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}\,\mathrm{d}\boldsymbol{x}\right|. \tag{8}$$

Next, we define the regularized Green's function by: $(\boldsymbol{G}, Q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$, solution of

$$-\Delta\boldsymbol{G} + \nabla Q = -\frac{\partial}{\partial x_j}(\delta_M \boldsymbol{e}_i), \qquad \operatorname{div}\boldsymbol{G} = 0, \tag{9}$$

where $\boldsymbol{e}_i$ is the ith unit canonical vector, and we define its Stokes projection $(\boldsymbol{G}_h, Q_h) \in X_h \times M_h$ by:

$$\int_\Omega \nabla\boldsymbol{G}_h : \nabla\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \int_\Omega Q_h \operatorname{div}\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\boldsymbol{G} : \nabla\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \int_\Omega Q \operatorname{div}\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} \quad \forall\boldsymbol{v}_h \in X_h, \tag{10}$$

$$\int_\Omega q_h \operatorname{div}\boldsymbol{G}_h \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall q_h \in M_h. \tag{11}$$

Then, we can show that

$$\int_\Omega \delta_M \frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \delta_M \frac{\partial \boldsymbol{u}_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \nabla\boldsymbol{u} : \nabla(\boldsymbol{G} - \boldsymbol{G}_h)\,\mathrm{d}\boldsymbol{x} + \int_\Omega p \operatorname{div}(\boldsymbol{G} - \boldsymbol{G}_h)\,\mathrm{d}\boldsymbol{x}, \tag{12}$$

and combined with (8), this implies indeed that the problem reduces to a uniform estimate for $\|\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^1(\Omega)}$. Finally, using Cauchy–Schwarz's inequality, we write:

$$\left\|\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\right\|_{L^1(\Omega)} \leqslant \left(\int_\Omega \sigma^\mu \left|\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\right|^2 \mathrm{d}\boldsymbol{x}\right)^{1/2} \left(\int_\Omega \sigma^{-\mu}\,\mathrm{d}\boldsymbol{x}\right)^{1/2}. \tag{13}$$

As we can easily prove that for $\mu = d + \lambda$, $0 < \lambda < 1$,

$$\int_\Omega \sigma^{-\mu}\,\mathrm{d}\boldsymbol{x} \leqslant \frac{C}{\lambda}(\kappa h)^{-\lambda}, \tag{14}$$

with a constant C independent of h and λ, this reduces now the problem to establishing the weighted error estimate for $\boldsymbol{G}_h$:

$$\left\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\right\|_{L^2(\Omega)} \leqslant C h^{\lambda/2}. \tag{15}$$

However, since (10) is a variational equation, the only straightforward way for introducing a weight into it is by inserting the weight into the test function. For this, we interpolate $\boldsymbol{G}$ with an interpolation operator P_h that *preserves the discrete divergence*, cf. Girault and Scott [8], we define the auxiliary function $\boldsymbol{\psi}$ by

$$\boldsymbol{\psi} = \sigma^\mu\big(P_h(\boldsymbol{G}) - \boldsymbol{G}_h\big), \tag{16}$$

and we use $\overline{P}_h(\boldsymbol{\psi})$ as test function, where $\overline{P}_h$ is a simplified version of P_h that takes advantage of the continuity of $\boldsymbol{\psi}$. This yields the following identity:

$$\int_\Omega \sigma^\mu \big|\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\big|^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla(\boldsymbol{G}-\boldsymbol{G}_h):\nabla\big[\big(\boldsymbol{G}-P_h(\boldsymbol{G})\big)\sigma^\mu\big]\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla(\boldsymbol{G}-\boldsymbol{G}_h):\nabla\big(\boldsymbol{\psi}-\overline{P}_h(\boldsymbol{\psi})\big)\,\mathrm{d}\boldsymbol{x}$$
$$- \int_\Omega \big(\nabla(\boldsymbol{G}-\boldsymbol{G}_h)(\boldsymbol{G}-\boldsymbol{G}_h)\big)\cdot\nabla\sigma^\mu\,\mathrm{d}\boldsymbol{x} + \int_\Omega (Q-Q_h)\,\mathrm{div}\big(\overline{P}_h(\boldsymbol{\psi})\big)\,\mathrm{d}\boldsymbol{x}. \tag{17}$$

All the subsequent steps are devoted to estimating the terms in the right-hand side of (17).

Weighted estimates. In view of the first term, we must derive a weighted estimate for the interpolation error $\nabla(\boldsymbol{G}-P_h(\boldsymbol{G}))$. This is the object of the *second step*, that establishes the weighted bounds:

$$\big\|\sigma^{\mu/2}\nabla_2\boldsymbol{G}\big\|_{L^2(\Omega)} + \big\|\sigma^{\mu/2}\nabla Q\big\|_{L^2(\Omega)} \leqslant C\kappa^{\mu/2}h^{\lambda/2-1}, \tag{18}$$

where ∇_k denotes the kth-order derivatives tensor. It is essentially based on two arguments: a duality argument for $\boldsymbol{G}$, similar to that used by [10] and [1], and a weighted inf–sup condition for Q, that applies [5] with the non-critical exponent $\alpha = -(\mu/2-1)$ utilizing $0<\lambda<1$. Let us remark here that a weighted estimate for the interpolation error of P_h also requires that P_h be quasi-local. For this, we refer to [8], where quasi-local interpolation operators are constructed for a large class of finite-elements.

Super-approximation. The second term in the right-hand side of (17) involves a weighted estimate for $\nabla(\boldsymbol{\psi}-\overline{P}_h(\boldsymbol{\psi}))$. More specifically, we prove that

$$\big\|\sigma^{-\mu/2}\nabla\big(\boldsymbol{\psi}-\overline{P}_h(\boldsymbol{\psi})\big)\big\|_{L^2(\Omega)} \leqslant C\big\|\sigma^{\mu/2-1}\big(P_h(\boldsymbol{G})-\boldsymbol{G}_h\big)\big\|_{L^2(\Omega)}, \tag{19}$$

with a constant C independent of h. Since $\boldsymbol{\psi}$ has the factor σ^μ, it is possible to see that (19) follows mainly from a "super-approximation" result that eliminates the highest-order derivative of $P_h(\boldsymbol{G})-\boldsymbol{G}_h$ in the right-hand side of the error bound. This is particularly technical when dealing with incomplete spaces of polynomials, used frequently with fluids. The *third step* is devoted to establishing this "super-approximation" result for the "mini-element", the Taylor–Hood finite elements and the Bernardi–Raugel element.

Further weighted estimates. This step is motivated by the last two terms in the right-hand side of (17). On the one hand, the third term has the bound:

$$\int_\Omega \big(\nabla(\boldsymbol{G}-\boldsymbol{G}_h)(\boldsymbol{G}-\boldsymbol{G}_h)\big)\cdot\nabla\sigma^\mu\,\mathrm{d}\boldsymbol{x} \leqslant \mu\big\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\big\|_{L^2(\Omega)}\big\|\sigma^{\mu/2-1}(\boldsymbol{G}-\boldsymbol{G}_h)\big\|_{L^2(\Omega)}.$$

On the other hand, the fourth term, i.e. the one involving the pressure, can be reduced essentially to two terms:

$$\int_\Omega \big(r_h(Q)-Q_h\big)\sigma^\mu\,\mathrm{div}\big(\boldsymbol{G}_h-P_h(\boldsymbol{G})\big)\,\mathrm{d}\boldsymbol{x}\ ,\ \int_\Omega (Q-Q_h)\big(\boldsymbol{G}_h-P_h(\boldsymbol{G})\big)\nabla\sigma^\mu\,\mathrm{d}\boldsymbol{x}, \tag{20}$$

where $r_h(Q)$ is an interpolant of Q. The first term in (20) is simpler because $\boldsymbol{G}_h-P_h(\boldsymbol{G})$ has discrete divergence zero. Thus we can insert an approximation of the product $(r_h(Q)-Q_h)\sigma^\mu$ into this term and we prove that

$$\int_\Omega \big(r_h(Q)-Q_h\big)\sigma^\mu\,\mathrm{div}\big(\boldsymbol{G}_h-P_h(\boldsymbol{G})\big)\,\mathrm{d}\boldsymbol{x}$$
$$\leqslant Ch\big\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\big\|_{L^2(\Omega)}\big\|\sigma^{\mu/2-1}\big(r_h(Q)-Q_h\big)\big\|_{L^2(\Omega)}. \tag{21}$$

Then the pressure factor in the right-hand side can be estimated by means of the weighted inf–sup condition with non-critical exponent $-(\mu/2-1)$, since $0<\lambda<1$, and we see that the factor h exactly compensates the factor h^{-1} in (18).

The second term in (20) is much more problematic because the obvious factorization, which after simplification gives

$$\int_\Omega (Q-Q_h)(\boldsymbol{G}-\boldsymbol{G}_h)\cdot\nabla\sigma^\mu\,\mathrm{d}\boldsymbol{x} \leqslant \mu\big\|\sigma^{\mu/2}(Q-Q_h)\big\|_{L^2(\Omega)}\big\|\sigma^{\mu/2-1}(\boldsymbol{G}-\boldsymbol{G}_h)\big\|_{L^2(\Omega)},$$

is useless as it requires the weighted inf–sup condition with exponent $-\mu/2$, i.e. beyond the admissible range. In order to stay within the non-critical range, we consider the factorization

$$\int_\Omega (Q - Q_h)(\boldsymbol{G} - \boldsymbol{G}_h) \cdot \nabla\sigma^\mu \,\mathrm{d}\boldsymbol{x} \leqslant \mu \left\|\sigma^{\mu/2-\varepsilon/2}(Q - Q_h)\right\|_{L^2(\Omega)} \left\|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\right\|_{L^2(\Omega)}, \tag{22}$$

where $\varepsilon = \lambda + \gamma$ for some small $\gamma > 0$. Thus, in view of these two terms, and since λ itself is also small, we are led to find an appropriate bound for

$$\left\|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\right\|_{L^2(\Omega)}, \tag{23}$$

for small $\varepsilon \geqslant 0$. We estimate it by means a duality argument that generalizes the argument of [10] and [1] for evaluating (23) with $\varepsilon = 0$. We prove first that

$$\left\|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\right\|_{L^2(\Omega)} \leqslant \frac{C_1}{\sqrt{\kappa}} \left\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\right\|_{L^2(\Omega)} + C_2\kappa^{\mu/2-1/2}h^{\lambda/2}, \tag{24}$$

and next that

$$\left\|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\right\|_{L^2(\Omega)} \leqslant C_3 \frac{(\kappa h)^{\varepsilon/2}}{\sqrt{\kappa}} \left(\left\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\right\|_{L^2(\Omega)} + C_4\kappa^{\mu/2}h^{\lambda/2}\right). \tag{25}$$

Observe that the factor $h^{\varepsilon/2}$ exactly compensates the $-\varepsilon/2$ in the exponent of the first factor of the right-hand side of (22).

Assembly. The parameter κ in (25), that is part of the weight (4), appears in the denominator multiplying $\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}$. Choosing κ sufficiently large, this term can be absorbed into the left-hand side of (17). The same happens with other similar terms occurring in the right-hand side of (17). This concludes the proof.

Several steps in this proof restrict the triangulation and the domain. Indeed, since σ is a function of the global mesh-size, the proofs of some estimates use a uniformly regular (or quasi-uniform) triangulation. This is also the case in [10] and [1]. However, relaxing, even partially, this restriction is not straightforward.

As far as the domain is concerned, the above duality argument restricts from the start the angles of $\partial\Omega$. Indeed, in view of the Sobolev imbedding

$$W^{2,r}(\Omega) \subset W^{1,\infty}(\Omega), \quad \text{for } r > d,$$

the angles must be such that the solution $(\boldsymbol{v}, q)$ of the Stokes problem

$$-\Delta\boldsymbol{v} + \nabla q = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega, \quad \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}, \tag{26}$$

belongs to $W^{2,r}(\Omega)^d \times W^{1,r}(\Omega)$ whenever $\boldsymbol{f}$ belongs to $L^r(\Omega)^d$ for some real number $r > d$. In two dimensions, this holds when Ω is convex, and r depends on the largest inner angle of $\partial\Omega$ (see Grisvard [9]). However, in three dimensions, convexity is not sufficient (see Dauge [4]): the largest inner dihedral angle of $\partial\Omega$ must be strictly less than $2\pi/3$, the precise value depending on r. This amount of regularity is essentially consistent with requiring that p and the gradient of $\boldsymbol{u}$ be bounded, in the sense that the restriction on the angles is the same. Thus our restrictions on the boundary are best possible consistent with our goal of providing error estimates for the approximation of p and the gradient of $\boldsymbol{u}$ in the maximum norm.

2. Sketch of the proof for the pressure

An estimate for the pressure must be obtained by duality because an L^∞ bound cannot be derived directly from the velocity, since the inf–sup condition is usually not valid in L^∞. Let $\boldsymbol{x}_M$ be a point in $\overline{\Omega}$ where $|p_h(\boldsymbol{x})|$ attains its maximum and associate with $|p_h(\boldsymbol{x}_M)|$ a function δ_M as in (8). Let $(\boldsymbol{G}, Q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ be the solution of

$$-\Delta \boldsymbol{G} + \nabla Q = 0, \quad \operatorname{div} \boldsymbol{G} = \delta_M - B, \tag{27}$$

where B is a fixed function of $\mathcal{D}(\Omega)$ such that $\int_\Omega B(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$. Then, we define $\boldsymbol{G}_h \in X_h$, the Stokes projection of $\boldsymbol{G}$, and its associated pressure $Q_h \in M_h$ by

$$\int_\Omega \nabla(\boldsymbol{G}_h - \boldsymbol{G}) : \nabla \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} + \int_\Omega (Q - Q_h) \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall \boldsymbol{v}_h \in X_h, \tag{28}$$

$$\int_\Omega q_h \operatorname{div}(\boldsymbol{G}_h - G)\,\mathrm{d}\boldsymbol{x} = 0 \quad \forall q_h \in M_h. \tag{29}$$

As we have now a $W^{1,\infty}$ bound for $\boldsymbol{u}_h$, we can easily reduce the proof of the second part of (3) to that of a uniform estimate for $\nabla(\boldsymbol{G}_h - \boldsymbol{G})$ in $L^1(\Omega)$. Since Eqs. (27) defining $(\boldsymbol{G}, Q)$ are similar to (9), this is much like the above proof.

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