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Differential Geometry

Dirac structures and paracomplex manifolds

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Abstract

We introduce the notion of a generalized paracomplex structure. This is a natural notion which unifies several geometric structures such as symplectic forms, paracomplex structures, and Poisson structures. We show that generalized paracomplex structures are in one-to-one correspondence with pairs of transversal Dirac structures on a smooth manifold. ***To cite this article: A. Wade, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Structures de Dirac et variétés paracomplexes. Nous introduisons la notion de structure paracomplexe généralisée. Il s'agit d'une notion naturelle qui unifie plusieurs structures géométriques telles que les formes symplectiques, les structures paracomplexes et les structures de Poisson. Nous montrons que, sur une variété différentiable, les structures paracomplexes généralisées sont en correspondance biunivoque avec les couples de structures de Dirac transverses. ***Pour citer cet article : A. Wade, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Les structures presque paracomplexes ont été introduites par Libermann en 1952 (voir [6]). Rappelons qu'une structure presque paracomplexe sur une variété différentiable de dimension paire est la donnée d'un opérateur $P : TM \to TM$ tel que $P^2 = \mathrm{id}$, $P \neq \pm\,\mathrm{id}$ et tel que les sous-fibrés, associés aux valeurs propres 1 et -1, ont le même rang. Si, en outre, la torsion de Nijenhuis de P est nulle alors nous disons que P est une structure paracomplexe. La géométrie paracomplexe a été étudiée par de nombreux mathématiciens. Nous renvoyons le lecteur à la référence [3] pour plus de détails sur ce sujet.

L'idée de cette Note est de considérer une structure presque paracomplexe non pas sur le fibré tangent d'une variété différentiable M mais sur la somme de Whitney $TM \oplus T^*M$. Ceci nous conduit à une notion qui unifie les structures presque paracomplexes, les formes symplectiques et les structures de Poisson.

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Soit M une variété différentiable de dimension n. Nous considérons le fibré vectoriel $TM \oplus T^*M$ muni de l'application bilinéaire symétrique définie par :

$$\langle X_1 + \alpha_1,\ X_2 + \alpha_2 \rangle = \frac{1}{2}(i_{X_2}\alpha_1 + i_{X_1}\alpha_2).$$

Le crochet de Courant sur $TM \oplus T^*M$ est défini par la formule suivante :

$$[X_1 + \alpha_1, X_2 + \alpha_2]_C = [X_1, X_2] + \left(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + \frac{1}{2}d(i_{X_2}\alpha_1 - i_{X_1}\alpha_2)\right),$$

pour tous $X_1+\alpha_1, X_2+\alpha_2 \in \Gamma(TM \oplus T^*M)$, où $\mathcal{L}_X$ est la dérivée de Lie dans la direction du champ de vecteur X. Afin de simplifier la notation, nous utiliserons $[\cdot,\cdot]$ au lieu de $[\cdot,\cdot]_C$.

Définition 0.1. Une *structure de Dirac* sur M est un sous-fibré L du fibré vectoriel $TM \oplus T^*M$ qui est isotrope maximal par rapport à $\langle\cdot,\cdot\rangle$ et dont l'espace $\Gamma(L)$ des sections différentiables est stable pour le crochet de Courant.

Étant donné un automorphisme $\mathcal{J}$ du fibré vectoriel $TM \oplus T^*M$, nous definissons :

$$N_{\mathcal{J}}(e_1, e_2) = [\mathcal{J}e_1, \mathcal{J}e_2] + \mathcal{J}^2[e_1, e_2] - \mathcal{J}[\mathcal{J}e_1, e_2] - \mathcal{J}[e_1, \mathcal{J}e_2],$$

pour tous $e_1, e_2 \in \Gamma(TM \oplus T^*M)$.

Définition 0.2. Soit $\mathcal{J}$ un automorphisme du fibré vectoriel $TM \oplus T^*M$. On dit que $\mathcal{J}$ est une *structure presque paracomplexe généralisée* si $\mathcal{J}^2 = \mathrm{id}$, $\mathcal{J} \neq \pm\,\mathrm{id}$ et si $\mathcal{J}$ est compatible avec l'application bilinéaire symétrique $\langle\cdot,\cdot\rangle$, c'est-à-dire

$$\langle \mathcal{J}e_1, e_2\rangle + \langle e_1, \mathcal{J}e_2\rangle = 0,$$

pour tous $e_1, e_2 \in \Gamma(TM \oplus T^*M)$. Si, en outre, le tenseur de Nijenhuis $N_{\mathcal{J}}$ est nul alors on dira que $\mathcal{J}$ est une structure paracomplexe généralisée.

Nous avons les résultats suivants :

Proposition 0.3. *Toute structure presque paracomplexe généralisée détermine deux sous-fibrés de $TM \oplus T^*M$ qui sont transverses et isotropes maximaux par rapport à l'application bilinéaire symétrique $\langle\cdot,\cdot\rangle$.*

Théorème 0.4. *Il y a une correspondance biunivoque entre les structures presque paracomplexes généralisées et les couples de structures de Dirac tranverses sur une variété différentiable.*

Dans cette Note, nous montrons que les structures de Poisson, les formes symplectiques et les structures produits sont des exemples de structures paracomplexes généralisées.

1. Introduction

Let M be an even-dimensional smooth manifold. An *almost paracomplex structure* on M is a $(1,1)$-tensor field M such that $P^2 = \mathrm{id}$, $P \neq \pm\,\mathrm{id}$, and the two eigenbundles corresponding to the eigenvalues 1 and -1 have the same rank. The Nijenhuis tensor of such a $(1,1)$-tensor field is given by

$$N_P(X, Y) = [PX, PY] + [X, Y] - P\big([PX, Y] + [X, PY]\big),$$

for any vector fields X, Y on M. In this Note, instead of considering $(1,1)$-tensor fields, we look at automorphisms $\mathcal{J}$ of the vector bundle $TM \oplus T^*M$ satisfying $\mathcal{J}^2 = \mathrm{id}$, $\mathcal{J} \neq \pm\,\mathrm{id}$. This leads us to the notion of a generalized paracomplex structure which unifies symplectic forms, paracomplex structures, and Poisson structures. It is known that Dirac structures (see [1] and [2]) generalize both symplectic forms and Poisson structures. But, here we present a new point of view, which is based on the definition of the Nijenhuis tensor for certain automorphisms of the Courant algebroid $TM \oplus T^*M$. We show the relationship between generalized paracomplex structures and Dirac structures.

2. Generalized almost paracomplex structures

Let M be an n-dimensional smooth manifold. The vector bundle $TM \oplus T^*M$ is endowed with the canonical symmetric bilinear operation given by

$$\langle X_1 + \alpha_1,\ X_2 + \alpha_2\rangle = \frac{1}{2}(i_{X_2}\alpha_1 + i_{X_1}\alpha_2).$$

Definition 2.1. An *almost Dirac structure* is a subbundle L of $TM \oplus T^*M$ which is maximal isotropic with respect to $\langle\cdot,\cdot\rangle$, i.e. $\dim L = \dim M = n$ and $\langle e_1, e_2\rangle = 0$, for any smooth sections $e_1, e_2 \in \Gamma(L)$.

Definition 2.2. A *generalized almost paracomplex structure* on M is a vector bundle automorphism $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ such that $\mathcal{J}^2 = \mathrm{id}$, $\mathcal{J} \neq \pm\,\mathrm{id}$, and $\mathcal{J}$ is orthogonal with respect to $\langle\cdot,\cdot\rangle$, i.e.,

$$\langle \mathcal{J}e_1, e_2\rangle + \langle e_1, \mathcal{J}e_2\rangle = 0, \quad \forall e_1, e_2 \in TM \oplus T^*M.$$

Such a map can be represented by a matrix of the form

$$\mathcal{J} = \begin{pmatrix} A & \Pi \\ \Omega & -A^* \end{pmatrix},$$

where Ω is a 2-form, and Π is a bivector field.

Proposition 2.3. *Let $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ be a generalized almost paracomplex structure on a smooth n-dimensional manifold M. Then the eigenbundles corresponding to 1 and -1 are almost Dirac subbundles of $TM \oplus T^*M$ which are transversal to each other.*

Proof. If L^+, L^- are the eigenbundles corresponding to 1 and -1 respectively, then $L^+ \cap L^- = \{0\}$. Moreover, L^+ and L^- are totally isotropic since $\mathcal{J}$ is orthogonal with respect to $\langle\cdot,\cdot\rangle$. Let $e_0 \in \Gamma(TM \oplus T^*M)$ be a smooth section such that $\langle e_0, \ell\rangle = 0$, for any $\ell \in \Gamma(L^+)$. For any $e \in \Gamma(TM \oplus T^*M)$,

$$e = \frac{1}{2}\big((e + \mathcal{J}e) + (e - \mathcal{J}e)\big), \quad \ell_1 = e + \mathcal{J}e \in \Gamma(L^+), \quad \text{and} \quad \ell_2 = e - \mathcal{J}e \in \Gamma(L^-).$$

$$\begin{aligned} 2\langle e_0 - \mathcal{J}e_0,\ e\rangle &= \langle e_0 - \mathcal{J}e_0, \ell_1 + \ell_2\rangle = \langle e_0, \ell_1 + \ell_2\rangle - \langle \mathcal{J}e_0, \ell_1 + \ell_2\rangle \\ &= \langle e_0, \ell_1\rangle + \langle e_0, \ell_2\rangle + \langle e_0, \ell_1\rangle - \langle e_0, \ell_2\rangle = 2\langle e_0, \ell_1\rangle = 0. \end{aligned}$$

This implies $\mathcal{J}e_0 = e_0$. Hence L^+ is maximal isotropic with respect to $\langle\cdot,\cdot\rangle$. Similarly, one can show that L^- is also maximal isotropic.

3. Integrability

Let M be a finite dimensional smooth manifold. The *Courant bracket* is an $\mathbb{R}$-bilinear operation on the space $\Gamma(TM \oplus T^*M)$ of smooth sections of $TM \oplus T^*M$ given by

$$[X_1+\alpha_1, X_2+\alpha_2]_C = [X_1, X_2] + \left(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + \frac{1}{2}d(i_{X_2}\alpha_1 - i_{X_1}\alpha_2)\right),$$

for any $X_1+\alpha_1, X_2+\alpha_2 \in \Gamma(TM \oplus T^*M)$, where $\mathcal{L}_X$ is the Lie derivation by X. For simplicity, we will use $[\cdot,\cdot]$ instead of $[\cdot,\cdot]_C$.

Definition 3.1. A Dirac structure on M is an *almost Dirac structure* (*or Dirac subbundle*) L of $TM \oplus T^*M$ such that the space $\Gamma(L)$ of sections of L is closed under the Courant bracket.

Given a vector bundle automorphism $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$, we can define the map $N_{\mathcal{J}}$ given by

$$N_{\mathcal{J}}(e_1, e_2) = [\mathcal{J}e_1, \mathcal{J}e_2] + \mathcal{J}^2[e_1, e_2] - \mathcal{J}[\mathcal{J}e_1, e_2] - \mathcal{J}[e_1, \mathcal{J}e_2],$$

for all $e_1, e_2 \in \Gamma(TM \oplus T^*M)$.

Definition 3.2. An automorphism $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ is *integrable* if $N_{\mathcal{J}}(e_1, e_2) = 0$, for all $e_1, e_2 \in \Gamma(TM \oplus T^*M)$. If $\mathcal{J}$ is an integrable generalized almost paracomplex structure then we simply call it a *generalized paracomplex structure*.

We should mention that there are similarities between the notion of generalized paracomplex structure and that of a generalized complex structure which was introduced in [5] and studied in [4] (even though generalized complex structures exist only on even-dimensional manifolds). In fact, this Note is inspired by works of Hitchin and Gualtieri.

Theorem 3.3. *A generalized almost paracomplex structure $\mathcal{J}$ is integrable if and only if the eigenbundles $L^\pm$ corresponding to the eigenvalues $\lambda = \pm 1$ are Dirac subbundles of $TM \oplus T^*M$.*

Proof. Suppose that $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ is a generalized paracomplex structure on M. Let L^+ be the eigenbundle corresponding to the eigenvalue $\lambda = 1$, that is,

$$\Gamma(L^+) = \{\mathcal{J}e + e\colon e \in \Gamma(TM \oplus T^*M)\}.$$

By Proposition 2.3, L^+ is an almost Dirac structure. Furthermore,

$$[\mathcal{J}e_1 + e_2, \mathcal{J}e_2 + e_2] = [\mathcal{J}e_1, \mathcal{J}e_2] + [e_1, e_2] + [\mathcal{J}e_1, e_2] + [e_1, \mathcal{J}e_2].$$

Clearly, $\Gamma(L^+)$ is closed under the Courant bracket if $N_{\mathcal{J}} = 0$. Similarly, if $N_{\mathcal{J}} = 0$ then $\Gamma(L^-)$ is closed under the Courant bracket. Conversely, suppose that both $\Gamma(L^+)$ and $\Gamma(L^-)$ are closed under the Courant bracket. Let $\Psi^+ : \Gamma(TM \oplus T^*M) \to \Gamma(L^+)$ and $\Psi^- : \Gamma(TM \oplus T^*M) \to \Gamma(L^-)$ be the projection maps defined by

$$\Psi^+(e) = \frac{1}{2}(e + \mathcal{J}e), \qquad \Psi^-(e) = \frac{1}{2}(e - \mathcal{J}e).$$

Since $\Gamma(L^+)$ and $\Gamma(L^-)$ are closed under the Courant bracket, we get

$$\Psi^-[\Psi^+e_1, \Psi^+e_2] = 0 \quad \text{and} \quad \Psi^+[\Psi^-e_1, \Psi^-e_2] = 0.$$

By a straightforward computation, one gets

$$N_{\mathcal{J}}(e_1, e_2) = 4\left(\Psi^-[\Psi^+e_1, \Psi^+e_2] + \Psi^+[\Psi^-e_1, \Psi^-e_2]\right).$$

There follows $N_{\mathcal{J}} = 0$. □

Theorem 3.4. *There is a one-to-one correspondence between generalized paracomplex structures on M and the set of pairs of transversal Dirac subbundles of $TM \oplus T^*M$.*

Proof. By Theorem 3.3, every generalized paracomplex structure determines a pair of transversal Dirac subbundles of $TM \oplus T^*M$. Conversely, suppose that (L^+, L^-) is a pair of transversal Dirac subbundles of $TM \oplus T^*M$. We set $\mathcal{J}_{|L^+} = \mathrm{id}$ and $\mathcal{J}_{|L^-} = -\mathrm{id}$ then $\mathcal{J}$ is a generalized paracomplex structure. □

4. Examples of generalized paracomplex structures

Example 1 (*The trivial generalized paracomplex structure*). Consider the vector bundle automorphism of $TM \oplus T^*M$ given by

$$\mathcal{J}_0 = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

The eigenbundles corresponding to the eigenvalues $\lambda = \pm 1$ are TM and T^*M, which are trivial Dirac subbundles.

Example 2 (*Symplectic structures*). Let M be a smooth manifold endowed with a non-degenerate 2-form Ω. Define the generalized almost paracomplex structure:

$$\mathcal{J}_\Omega = \begin{pmatrix} 0 & \Omega^{-1} \\ \Omega & 0 \end{pmatrix}.$$

The eigenbundles corresponding to the eigenvalues $\lambda = \pm 1$ are

$$L^+ = \{X + i_X\Omega\colon X \in TM\} \quad \text{and} \quad L^- = \{X - i_X\Omega\colon X \in TM\}.$$

By Theorem 3.3, $\mathcal{J}_\Omega$ is integrable if and only if L^+ and L^- are Dirac subbundles. It is known (see [2]) that these subbundles are Dirac if and only if Ω is closed (i.e., $\mathrm{d}\Omega = 0$). Hence, $\mathcal{J}_\Omega$ is integrable if and only if Ω is a symplectic 2-form on M.

Example 3 (*Poisson structures*). Let π be a bivector on a smooth manifold M. Consider the generalized almost paracomplex structure given by

$$\mathcal{J}_\pi = \begin{pmatrix} Id & -2\pi \\ 0 & -Id \end{pmatrix}.$$

The eigenbundle corresponding to the eigenvalue $\lambda = -1$ is

$$L^- = \{e - \mathcal{J}_\pi e\colon e \in TM \oplus T^*M\} = \{\pi\alpha + \alpha\colon \alpha \in T^*M\}.$$

The eigenbundle corresponding to the eigenvalue $\lambda = 1$ is

$$L^+ = \{e + \mathcal{J}_\pi e\colon e \in TM \oplus T^*M\} = TM.$$

By Theorem 3.3, $\mathcal{J}_\pi$ is integrable if and only if L^+ and L^- are Dirac subbundles of $TM \oplus T^*M$. But, L^- is a Dirac subbundle if and only if the Schouten–Nijenhuis bracket $[\pi, \pi]_S = 0$. Obviously, L^+ is a Dirac subbundle. Therefore, $\mathcal{J}_\pi$ is integrable if and only if π is a Poisson structure on M.

Example 4 (*Product structures*). An almost product structure is a weakened version of the notion of an almost paracomplex structure. More precisely, an *almost product structure* is a $(1,1)$-tensor field P such that $P^2 = \mathrm{id}$. To such a structure, we associate the generalized almost paracomplex structure on M given

$$\mathcal{J} = \begin{pmatrix} P & 0 \\ 0 & -P^* \end{pmatrix}.$$

The corresponding almost Dirac structures are:

$$L^+ = \mathrm{Ker}(P - \mathrm{id}) \oplus \mathrm{Ann}\big(\mathrm{Ker}(P - \mathrm{id})\big),$$
$$L^- = \mathrm{Ker}(P + \mathrm{id}) \oplus \mathrm{Ann}\big(\mathrm{Ker}(P + \mathrm{id})\big).$$

The notation "Ann" stands for the annihilator of a subbundle of TM. It is easy to check that $\mathcal{J}$ is integrable if and only if the Nijenhuis tensor of P vanishes.

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