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Partial Differential Equations

General entropy equations for structured population models and scattering

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Abstract

We consider several structured population models (age structured, size structured, maturity structured) and the general scattering equation. These models are not conservation laws, nevertheless, we show that they admit a common relative entropy structure which uses the first eigenelements of the problem. In case of scattering, it is more general than the usual 'detailed balance principle'. Three types of consequences are deduced from this entropy structure: a priori bounds, large time convergence to the steady state and in some cases, exponential rates of convergence. ***To cite this article: P. Michel et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Un principe général d'entropie pour les modèles de population structurés et les équations de scattering. Nous considérons divers modèles de populations structurées (en âge, en taille ou en maturité) et aussi l'équation de scattering. Ces modèles ne sont pas conservatifs, néammoins nous montrons qu'ils vérifient tous une structure d'entropie relative commune qui utilise les premiers éléments propres du problème et qui, dans le cas du scattering, généralise le « principe d'équilibre en détail » habituel. Trois types de conséquences découlent de cette structure entropique : des estimations a priori, la convergence en temps grand vers un état stationnaire et parfois des taux exponentiels de convergence. ***Pour citer cet article : P. Michel et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Un des objets de modélisation classique en biomathématiques consiste en la notion de « population structurée » (en âge, taille, maturation). On considère alors la fonction de densité $n(t,x)$ d'individus ou de cellules de para-

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mètre $x \geqslant 0$. Plusieurs équations sont utilisées pour décrire l'évolution de cette densité, voir (3), (4), ainsi que les références [15,2,19,12].

Par exemple le modèle structuré en taille donne lieu à l'équation :

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + b(x)n(t,x) = \int_0^1 \beta(\frac{x}{\sigma},\sigma)n(t,\frac{x}{\sigma})\frac{\mathrm{d}\sigma}{\sigma^2}, \quad t \geqslant 0,\ x \geqslant 0, \\ n(t,x=0) = 0. \end{cases} \tag{1}$$

Pour ce modèle, ainsi que deux autres présentés ci-dessous, on introduit un triplet (λ, N, ϕ) avec λ la première valeur propre de l'opérateur stationnaire, $N \geqslant 0$ le vecteur propre associé et $\phi \geqslant 0$ le vecteur propre du problème dual. Ces trois modèles vérifient le principe de décroissance d'entropie relative suivant :

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \phi(x)N(x)H\left(\frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}\right)\mathrm{d}x = D^H(t) \leqslant 0, \tag{2}$$

pour toute fonction convexe H avec des taux de dissipation D^H donnés ci-dessous.

On en déduit trois types de conséquences. D'abord, on obtient directement que la fonction $\frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}$ reste bornée au cours du temps par $\|\frac{n(0,x)}{N(x)}\|_{L^\infty}$, ou bien dans les L^p avec les poids $\phi(x)N(x)$. Ensuite, pour ces modèles, l'annulation de la quantité $D^H(n)$ pour n solution de l'équation d'évolution, induisent que la fonction $\frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}$ est constante. On en déduit assez généralement la convergence en temps long de la solution vers un état stationnaire. En dernier lieu, on peut essayer d'en déduire un taux exponentiel de convergence vers l'état stationnaire ce qui n'est réalisé que partiellement [13,15,16,18].

Prenons par exemple le modèle structuré en taille (1) avec $\beta(\frac{x}{\sigma},\sigma)$ ayant un support en σ contenant un intervalle non-vide $]r_0, r_1[$ de $]0,1[$. Lorsque l'on prend $H(m) = m^2$, on a alors :

$$D^H_{\mathrm{size}} = -\int_0^1\int_0^\infty \phi(x)N\left(\frac{x}{\sigma}\right)\beta\left(\frac{x}{\sigma},\sigma\right)\left[\left(\frac{n(t,x/\sigma)\,\mathrm{e}^{\lambda t}}{N(x/\sigma)} - \frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}\right)\right]^2\frac{\mathrm{d}\sigma}{\sigma^2}\,\mathrm{d}x \leqslant 0.$$

Dans ce cas, dès que $D^H_{\mathrm{size}}(m)$ est égal à zero on trouve que la quantité $\frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}$ est constante. On peut alors prouver directement que la distribution $n\,\mathrm{e}^{\lambda t}$ converge dans $L^2(N\phi\,\mathrm{d}x)$ vers un état stationnaire. En général l'argument nécessite d'utiliser aussi l'équation car $D^H(t) = 0$ ne permet pas de conclure que $n(t,x)\,\mathrm{e}^{\lambda t} = \mu(t)N(x)$, c'est le cas pour la mitose $\beta(y,\sigma) = b(y)\delta(\sigma = \frac{1}{2})$.

1. The models

Structured population equations are a classical topic in mathematical biology which cover a variety of models which have been studied independently and exhibit interesting mathematical structures being, generally based on transport equations with 'low order' regularizing terms. General references on the topic of structured population dynamics are [2,15]. The most classical model (derived by McKendrick in 1927) is the *age structured* model which describes the evolution of a density $n(t,x)$ of population with age $x > 0$

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + d(x)n(t,x) = \frac{\partial}{\partial x}\left[\nu(x)\frac{\partial n(t,x)}{\partial x}\right], \quad t \geqslant 0,\ x \geqslant 0, \\ n(t,x=0) - \nu(0)\frac{\partial n(t,x=0)}{\partial x} = \int_{\{x \geqslant 0\}} b(y)n(t,y)\,\mathrm{d}y. \end{cases} \tag{3}$$

Here $d(x) \geqslant 0$ is the death rate, the population gives newborns, i.e., of age 0, with a rate $b(\cdot) \geqslant 0$ and $\nu(x) \geqslant 0$ is a diffusion used as a simple model for various effects.

Another classical equation (usually called fragmentation or cell-division equation) consists in structuring the population by size x of the individuals (or DNA content for cells, see [3]). The model then is based on the equation

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + b(x)n(t,x) = \int_0^1 \beta(\frac{x}{\sigma},\sigma)n(t,\frac{x}{\sigma})\frac{\mathrm{d}\sigma}{\sigma^2}, \quad t \geqslant 0,\ x \geqslant 0, \\ n(t,x=0) = 0. \end{cases} \tag{4}$$

The main difference compared to the first model is that cell-division occurs with a rate $\beta(\cdot,\cdot) \geqslant 0$ which expresses the repartition of sizes in $(\sigma x, (1-\sigma)x)$ after mitosis and thus $\beta(y,\sigma) = \beta(y, 1-\sigma)$ and finally $b(y) := \int_0^1 \beta(y,\sigma)\,\mathrm{d}\sigma$. The special case of equal size $x/2$ after division is obtained when $\beta(y,\sigma) = b(y)\delta(\sigma = \frac{1}{2})$.

Various other models have been proposed which are of kinetic nature and are structured with an additional 'maturation' variable which is the speed of progression in the variable x. We refer to Lebowitz and Rubinow [12] and Rotenberg [19] for such models. They share some similarity with the classical scattering equation that we choose in order to show the extent of our theory,

$$\frac{\partial}{\partial t} n(t,x) + k_T(x) n(t,x) = \int_{\mathbb{R}^d} K(x,y) n(t,y)\,\mathrm{d}y. \tag{5}$$

Here $0 \leqslant k_T(\cdot) \in L^\infty(\mathbb{R}^d)$ and $0 \leqslant K(x,y) \in L^1 \cap L^\infty(\mathbb{R}^d)$ and especially we aim to treat the non-symmetric case as motivated by [8,14,4].

These equations are completed with a Cauchy data denoted by $u^0(x)$. In practical applications, as cell-cycle, several types of such models are coupled [1,3,5].

2. Common relative entropy principle

In order to state the entropy structure behind these models, we need the existence of first eigenelements ($\lambda \in \mathbb{R}$, $N(x) \geqslant 0$, $\phi(x) \geqslant 0$) where λ is the first eigenvalue, N the corresponding eigenvector and ϕ the dual eigenvector to the stationary equation that we always normalize as

$$\int N(x) = 1, \qquad \int N(x)\phi(x) = 1. \tag{6}$$

Namely

(i) for age structured equation (3),

$$\begin{cases} \frac{\partial}{\partial x} N(x) + [\lambda + d(x)] N(x) = \frac{\partial}{\partial x}\left[\nu(x) \frac{\partial N(x)}{\partial x}\right], & x \geqslant 0,\\ N(x=0) - \nu(0)\frac{\partial N(x=0)}{\partial x} = \int_{\{y \geqslant 0\}} b(y) N(y)\,\mathrm{d}y, \end{cases} \tag{7}$$

$$\begin{cases} -\frac{\partial}{\partial x}\phi(x) + [\lambda + d(x)]\phi(x) = b(x)\phi(0) + \frac{\partial}{\partial x}\left[\nu(x)\frac{\partial \phi(x)}{\partial x}\right], & x \geqslant 0,\\ \nu(0)\frac{\partial}{\partial x}\phi(x=0) = 0, \end{cases} \tag{8}$$

(ii) for size structured equation (4),

$$\begin{cases} \frac{\partial}{\partial x} N(x) + [\lambda + b(x)] N(x) = \int_0^1 \beta\left(\frac{x}{\sigma},\sigma\right) N\left(\frac{x}{\sigma}\right)\frac{\mathrm{d}\sigma}{\sigma^2}, & x \geqslant 0,\\ N(x=0) = 0, \end{cases} \tag{9}$$

$$\frac{\partial}{\partial x}\phi(x) - [\lambda + b(x)]\phi(x) = -\int_0^1 \beta(x,\sigma)\phi(\sigma x)\frac{\mathrm{d}\sigma}{\sigma}, \quad x \geqslant 0, \tag{10}$$

(iii) for scattering (5),

$$[\lambda + k_T(x)] N(x) = \int_{\mathbb{R}^d} K(x,y) N(y)\,\mathrm{d}y, \tag{11}$$

$$[\lambda + k_T(x)]\phi(x) = \int_{\mathbb{R}^d} K(y,x)\phi(y)\,\mathrm{d}y. \tag{12}$$

Existence of these first positive eigenvalues requires specific assumptions in order to apply the Krein–Rutman theorem or variants (see [7]). We refer to [16] for these assumptions in this general context and just point out specific cases.

(i) For age structured model when $\nu = 0$, it is well known that, an assumption on (d, b) is necessary because λ, ϕ and N can be computed with explicit formula. For instance $d(x) \to \infty$ as $x \to \infty$ is enough when $b \in L^1 + L^\infty(\mathbb{R}^+)$.

(ii) For size-structured population with $\beta = b(\cdot)\delta(\sigma = \frac{1}{2})$ it is enough to assume that b is upper bounded and lower bounded away form 0 (see [18]); if b vanishes enough periodic solutions may exist [11]. There is a general class of β where an explicit solution N is known under the form of a series [16].

(iii) For scattering it holds true (with $\lambda = 0$, $\phi = 1$) under the assumption of *detailed balance* (natural for Boltzmann equation, see [13] and the references therein)

$$K(x,y)N(y) = K(y,x)N(x), \qquad k_T(x) = \int_{\mathbb{R}^d} K(y,x)\,\mathrm{d}y.$$

Theorem 2.1. *Consider Eqs.* (3), (4) *or* (5) *and suppose the first eigenelements* (λ, N, ϕ) *exists. Then we have, for all convex function* $H(\cdot)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \phi(x)N(x)H\left(\frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}\right)\mathrm{d}x = D^H(t) \leqslant 0, \tag{13}$$

where the entropy dissipation is given below for each model.

This entropy inequality is remarkable because the dual operator (via ϕ) compensates for non-symmetry in the operator. Such a dual operator is known to symmetrize elliptic (parabolic) operators but the connection with entropy for hyperbolic models seems to be new. It was first noticed in [17] for age structured models. Then we have respectively:

For the age strructured model $D^H_{\mathrm{age}} = D_1 + D_2$ with

$$D_1 = \phi(0)\int\left[H\left(\frac{n(t,0)}{N(0)}\right) - H\left(\frac{n(t,x)}{N(x)}\right) + H'\left(\frac{n(t,0)}{N(0)}\right)\left(\frac{n(t,x)}{N(x)} - \frac{n(t,0)}{N(0)}\right)\right]b(x)N(x)\,\mathrm{d}x,$$

$$D_2 = -\int_{\{x\geqslant 0\}} \nu(x)\left(\frac{\partial}{\partial x}\frac{\mathrm{e}^{\lambda t}n(t,x)}{N(x)}\right)^2 H''\left(\frac{\mathrm{e}^{\lambda t}n(t,x)}{N(x)}\right).$$

For the size structured model we find

$$D^H_{\mathrm{size}} = \int_0^1\int_{\{x\geqslant 0\}} \phi(x)N\left(\frac{x}{\sigma}\right)\beta\left(\frac{x}{\sigma},\sigma\right)\left[H'\left(\frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}\right)\left(\frac{n(t,x/\sigma)\,\mathrm{e}^{\lambda t}}{N(x/\sigma)} - \frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}\right)\right.$$
$$\left. - H\left(\frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}\right) + H\left(\frac{n(t,x/\sigma)\,\mathrm{e}^{\lambda t}}{N(x/\sigma)}\right)\right]\frac{\mathrm{d}\sigma}{\sigma^2}\,\mathrm{d}x$$

and for scattering

$$D^H_{\mathrm{scat}} = \int_{\mathbb{R}^d} K(x,y)N(y)\phi(x)H'\left(\frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}\right)\left(\frac{n(t,y)\,\mathrm{e}^{\lambda t}}{N(y)} - \frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}\right)\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant \int_{\mathbb{R}^d} K(x,y)N(y)\phi(x)\left[H\left(\frac{n(t,y)\,\mathrm{e}^{\lambda t}}{N(y)}\right) - H\left(\frac{n(t,x)\,\mathrm{e}^{\lambda t}}{N(x)}\right)\right]\mathrm{d}x\,\mathrm{d}y = 0.$$

3. Large time convergence to a steady state

Several properties follow classically from this entropy structure that allow one to derive a priori bounds on solutions and their convergence to a steady state.

To derive a priori bounds is a classical matter with the choice of an entropy $H(m) = |m|^p$ or $H(m) = (m-k)_+^2$ and this gives the

Lemma 3.1.

$$\int \phi(x)N(x)\left(\frac{|n(t,x)|\,\mathrm{e}^{\lambda t}}{N(x)}\right)^p \mathrm{d}x \leqslant \int \phi(x)N(x)\left(\frac{|n^0(x)|}{N(x)}\right)^p \mathrm{d}x, \quad 1 \leqslant p < \infty,$$

$$\inf_y \frac{n^0(y)}{N(y)} N(x) \leqslant n(t,x)\,\mathrm{e}^{\lambda t} \leqslant \sup_y \frac{n^0(y)}{N(y)} N(x).$$

The long time convergence is also a general feature which only uses the following property of the entropy dissipation term: for all the above models, for a strictly convex entropy H we have the

Theorem 3.2. *Consider Eqs.* (3), (4) *or* (5), *suppose the first eigenelements* (λ, N, ϕ) *exist and that the initial data satisfy* $\int \phi(x)|n^0(x)|\,\mathrm{d}x < \infty$. *Then we have,*

$$\left\|\phi\left[n(t,x)\,\mathrm{e}^{\lambda t} - \langle n^0\rangle N\right]\right\|_{L^1} \to 0 \quad \textit{as } t \to \infty,$$

where $\langle n^0\rangle = \int \phi n^0$.

This result is based on the entropy dissipation. After extraction of subsequences, we obtain a function that satisfies the equation and $D^H(t) = 0$ and this implies that $n(t,x)\,\mathrm{e}^{\lambda t} = \mu N(x)$. Usually the equality $D^H(t) = 0$ is not enough to conclude that $n(t,x)\,\mathrm{e}^{\lambda t} = \mu(t)N(x)$ and the equation has to be used again. For instance this is enough in the age structured model or for the size structured model (4) when the σ support of $\beta(y,\sigma)$ contains an interval $]r_0, r_1[$ (see the french version for additional details).

Rates of convergence rely on the control of entropy dissipation by the entropy itself and will be treated in a forthcoming paper. In these hyperbolic situations where compactness is not built in, this kind of control is an active subject see for instance [6,9,20,21]. Let us quote some known results

(i) (Age structured) With compactness assumptions (support of b bounded) compactness methods (and Laplace transform) are known to prove exponential rates of convergence (see [15]). This also holds true under the assumption $b(x) \geqslant \mu\phi(x)$ (then μ is the rate) (see [17]).
(ii) (Size structured) This is known for $b(x)$ constant and small perturbation of constants (see [18,16]) and nonlinear related models have been treated (see [10] and the references therein).
(iii) (Scattering) See [13] and the references therein for the case with $\lambda = 0, \phi = 1$.

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