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Mathematical Problems in Mechanics/Differential Geometry

An estimate of the H^1-norm of deformations in terms of the L^1-norm of their Cauchy–Green tensors

Philippe G. Ciarlet [a], Cristinel Mardare [b]

[a] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*
[b] *Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4, place Jussieu, 75005 Paris, France*

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Abstract

Let Ω be a bounded open connected subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary and let $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ be a deformation of the set $\overline{\Omega}$ satisfying $\det \nabla\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$. It is established that there exists a constant $C(\boldsymbol{\Theta})$ with the following property: for each deformation $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfying $\det \nabla\boldsymbol{\Phi} > 0$ a.e. in Ω, there exist an $n \times n$ rotation matrix $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta})$ and a vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi}, \boldsymbol{\Theta})$ in $\mathbb{R}^n$ such that

$$\left\|\boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R}\boldsymbol{\Theta})\right\|_{\boldsymbol{H}^1(\Omega)} \leqslant C(\boldsymbol{\Theta}) \left\|\nabla\boldsymbol{\Phi}^{\mathrm{T}}\nabla\boldsymbol{\Phi} - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\right\|_{\boldsymbol{L}^1(\Omega)}^{1/2}.$$

The proof relies in particular on a fundamental 'geometric rigidity lemma', recently proved by G. Friesecke, R.D. James, and S. Müller. ***To cite this article: P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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Résumé

Un majorant de la norme H^1 des déformations en fonction de la norme L^1 de leurs tenseurs de Cauchy–Green. Soit Ω un ouvert borné connexe de $\mathbb{R}^n$ à frontière lipschitzienne et soit $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ une déformation de l'ensemble $\overline{\Omega}$ satisfaisant dét $\nabla\boldsymbol{\Theta} > 0$ dans $\overline{\Omega}$. On établit l'existence d'une constante $C(\boldsymbol{\Theta})$ ayant la propriété suivante : quelle que soit la déformation $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfaisant dét $\nabla\boldsymbol{\Phi} > 0$ p.p. dans Ω, il existe une matrice $n \times n$ de rotation $\boldsymbol{R}$ et un vecteur $\boldsymbol{b} \in \mathbb{R}^n$ tels que

$$\left\|\boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R}\boldsymbol{\Theta})\right\|_{\boldsymbol{H}^1(\Omega)} \leqslant C(\boldsymbol{\Theta}) \left\|\nabla\boldsymbol{\Phi}^{\mathrm{T}}\nabla\boldsymbol{\Phi} - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\right\|_{\boldsymbol{L}^1(\Omega)}^{1/2}.$$

La démonstration repose en particulier sur un « lemme de rigidité géométrique » fondamental, récemmment établi par G. Friesecke, R.D. James, et S. Müller. ***Pour citer cet article : P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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E-mail addresses: mapgc@cityu.edu.hk (P.G. Ciarlet), mardare@ann.jussieu.fr (C. Mardare).

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1. Notations and other preliminaries

All spaces, matrices, etc., are real. The symbols $\mathbb{M}^n$, $\mathbb{S}^n$, $\mathbb{S}^n_>$, and $\mathbb{O}^n_+$ respectively designate the sets of all square matrices of order n, of all symmetric matrices of order n, of all positive-definite symmetric matrices of order n, and of all orthogonal matrices $\boldsymbol{Q}$ of order n with $\det \boldsymbol{Q} = 1$. A matrix $\boldsymbol{Q} \in \mathbb{O}^n_+$ will be called a *rotation*.

The Euclidean norm of a vector $\boldsymbol{b} \in \mathbb{R}^n$ is denoted $|\boldsymbol{b}|$ and $|\boldsymbol{A}| := \sup_{|\boldsymbol{b}|=1} |\boldsymbol{A}\boldsymbol{b}|$ denotes the spectral norm of a matrix $\boldsymbol{A} \in \mathbb{M}^n$. The Euclidean and spectral norms are invariant under rotations, in the sense that $|\boldsymbol{b}| = |\boldsymbol{Q}\boldsymbol{b}|$ and $|\boldsymbol{A}| = |\boldsymbol{Q}\boldsymbol{A}| = |\boldsymbol{A}\boldsymbol{Q}|$ for all rotations $\boldsymbol{Q} \in \mathbb{O}^n_+$.

Let Ω be an open subset of $\mathbb{R}^n$. Given any matrix-valued mapping $\boldsymbol{F} \in L^2(\Omega; \mathbb{M}^n)$, we let

$$\|\boldsymbol{F}\|_{L^2(\Omega;\mathbb{M}^n)} := \left\{ \int_\Omega \left|\boldsymbol{F}(x)\right|^2 \mathrm{d}x \right\}^{1/2},$$

and, given any vector-valued mapping $\boldsymbol{\Theta} \in H^1(\Omega; \mathbb{R}^n)$, we let

$$\|\boldsymbol{\Theta}\|_{H^1(\Omega;\mathbb{R}^n)} := \left\{ \int_\Omega \left(\left|\boldsymbol{\Theta}(x)\right|^2 + \left|\nabla\boldsymbol{\Theta}(x)\right|^2\right) \mathrm{d}x \right\}^{1/2},$$

where $\nabla\boldsymbol{\Theta}(x) \in \mathbb{M}^n$ denotes the gradient matrix of the mapping $\boldsymbol{\Theta}$ at x. *These norms are thus also invariant under rotations*, in the sense that $\|\boldsymbol{F}\|_{L^2(\Omega;\mathbb{M}^n)} = \|\boldsymbol{Q}\boldsymbol{F}\|_{L^2(\Omega;\mathbb{M}^n)} = \|\boldsymbol{F}\boldsymbol{Q}\|_{L^2(\Omega;\mathbb{M}^n)}$ and $\|\boldsymbol{\Theta}\|_{H^1(\Omega;\mathbb{R}^n)} = \|\boldsymbol{Q}\boldsymbol{\Theta}\|_{H^1(\Omega;\mathbb{R}^n)}$ for all rotations $\boldsymbol{Q} \in \mathbb{O}^n_+$.

In this Note, the space $\mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ is defined as that consisting of all vector-valued functions $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ that, together with their partial derivatives of the first order, possess continuous extentions to the closure $\overline{\Omega}$ of Ω, and the definition of a *bounded open set with a Lipschitz-continuous boundary* is the usual one, as found for instance in Nečas [14], Adams [1], or Grisvard [10].

2. A key inequality

The following theorem is the main result of this Note.

Theorem 2.1. *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ satisfying* $\det \nabla\boldsymbol{\Theta} > 0$ *in $\overline{\Omega}$, there exists a constant $C(\boldsymbol{\Theta})$ with the following property*: *given any mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfying* $\det \nabla\boldsymbol{\Phi} > 0$ *a.e. in Ω, there exist a vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ and a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$\left\|\boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R}\boldsymbol{\Theta})\right\|_{H^1(\Omega;\mathbb{R}^n)} \leqslant C(\boldsymbol{\Theta}) \left\|\nabla\boldsymbol{\Phi}^{\mathrm{T}}\nabla\boldsymbol{\Phi} - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\right\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

In this Note, we only give the proof of Theorem 2.1 under the additional assumption that the mapping $\boldsymbol{\Theta}$ is *injective* in $\overline{\Omega}$. The proof in the general case, which is substantially more technical and relies on a methodology reminiscent to that proposed in Ciarlet and Laurent [5], is found in Ciarlet and Mardare [8].

The proof of Theorem 2.1 in this special case is broken into those of four lemmas.

Lemma 2.2. *Let a matrix $\boldsymbol{F} \in \mathbb{M}^n$ be such that* $\det \boldsymbol{F} > 0$. *Then*

$$\operatorname{dist}\left(\boldsymbol{F}, \mathbb{O}^n_+\right) := \inf_{\boldsymbol{Q}\in\mathbb{O}^n_+} |\boldsymbol{F} - \boldsymbol{Q}| \leqslant \left|\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right|^{1/2}.$$

Proof. It is known that

$$\operatorname{dist}\left(\boldsymbol{F}, \mathbb{O}^n_+\right) = \left|\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\right)^{1/2} - \boldsymbol{I}\right|.$$

Let $0 < v_1 \leqslant v_2 \leqslant \cdots \leqslant v_n$ denote the singular values of the matrix $\boldsymbol{F}$. Then

$$\begin{aligned}\left|\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\right)^{1/2} - \boldsymbol{I}\right| &= \max\left\{|v_1 - 1|, |v_n - 1|\right\} \\ &\leqslant \max\left\{\left|v_1^2 - 1\right|^{1/2}, \left|v_n^2 - 1\right|^{1/2}\right\} = \left|\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}\right|^{1/2}. \quad \square\end{aligned}$$

Lemma 2.3. *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Then there exists a constant $\Lambda(\Omega)$ with the following property: given any mapping $\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n)$ satisfying* $\det\nabla\boldsymbol{\Phi} > 0$ *a.e. in Ω, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}) \in \mathbb{O}^n_+$ such that*

$$\|\nabla\boldsymbol{\Phi} - \boldsymbol{R}\|_{L^2(\Omega;\mathbb{M}^n)} \leqslant \Lambda(\Omega)\left\|\nabla\boldsymbol{\Phi}^{\mathrm{T}}\nabla\boldsymbol{\Phi} - \boldsymbol{I}\right\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Proof. By the '*geometric rigidity lemma*' of Friesecke, James and Müller [9, Theorem 3.1], there exists a constant $\Lambda(\Omega)$ depending only on the set Ω with the following property: for each $\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n)$, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}) \in \mathbb{O}^n_+$ such that

$$\|\nabla\boldsymbol{\Phi} - \boldsymbol{R}\|_{L^2(\Omega;\mathbb{M}^n)} \leqslant \Lambda(\Omega)\left\|\operatorname{dist}\left(\nabla\boldsymbol{\Phi}, \mathbb{O}^n_+\right)\right\|_{L^2(\Omega)}.$$

If in addition the mapping $\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n)$ satisfies $\det\nabla\boldsymbol{\Phi} > 0$ a.e. in Ω, then Lemma 2.2 implies that

$$\operatorname{dist}\left(\nabla\boldsymbol{\Phi}(x), \mathbb{O}^n_+\right) \leqslant \left|\nabla\boldsymbol{\Phi}(x)^{\mathrm{T}}\nabla\boldsymbol{\Phi}(x) - \boldsymbol{I}\right|^{1/2}$$

for almost all $x \in \Omega$. Hence

$$\left\|\operatorname{dist}\left(\nabla\boldsymbol{\Phi}, \mathbb{O}^n_+\right)\right\|_{L^2(\Omega)} \leqslant \left\|\nabla\boldsymbol{\Phi}^{\mathrm{T}}\nabla\boldsymbol{\Phi} - \boldsymbol{I}\right\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}. \quad \square$$

Lemma 2.4. *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Given any injective mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ satisfying* $\det\nabla\boldsymbol{\Theta} > 0$ *in $\overline{\Omega}$, there exists a constant $c(\boldsymbol{\Theta})$ with the following property: given any mapping $\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n)$ satisfying* $\det\nabla\boldsymbol{\Phi} > 0$ *a.e. in Ω, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi},\boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$\left\|\nabla\boldsymbol{\Phi} - \boldsymbol{R}\nabla\boldsymbol{\Theta}\right\|_{L^2(\Omega;\mathbb{M}^n)} \leqslant c(\boldsymbol{\Theta})\left\|\nabla\boldsymbol{\Phi}^{\mathrm{T}}\nabla\boldsymbol{\Phi} - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\right\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Proof. Since the boundary of Ω is Lipschitz-continuous, any mapping $\boldsymbol{\Theta}$ in the space $\mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ as defined in Section 1 can be extended to a mapping $\boldsymbol{\Theta}^\flat$ in the space $\mathcal{C}^1(\mathbb{R}^n;\mathbb{R}^n)$ (for a proof, see, e.g., Ciarlet and Mardare [6, Theorem 4.2], where this property is derived from the extension theorem of Whitney [16] combined with *ad hoc* Taylor formulas along paths). Moreover, since $\det\nabla\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$ and Ω is bounded, there exists a connected open subset $\Omega^\sharp$ containing $\overline{\Omega}$ such that the restriction $\boldsymbol{\Theta}^\sharp \in \mathcal{C}^1(\Omega^\sharp;\mathbb{R}^n)$ to $\Omega^\sharp$ of such an extension $\boldsymbol{\Theta}^\flat$ satisfies $\det\nabla\boldsymbol{\Theta}^\sharp > 0$ in $\Omega^\sharp$.

Consequently, the set $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$ is also a bounded connected open subset of $\mathbb{R}^n$ whose boundary $\boldsymbol{\Theta}(\partial\Omega) = \boldsymbol{\Theta}^\sharp(\partial\Omega)$ is Lipschitz-continuous. Besides, the inverse mapping $\widehat{\boldsymbol{\Theta}} : \{\widehat{\Omega}\}^- \to \overline{\Omega}$ of $\boldsymbol{\Theta}$ belongs to the space $\mathcal{C}^1(\{\widehat{\Omega}\}^-;\mathbb{R}^n)$, since each point of the boundary of $\widehat{\Omega}$ possesses a neighborhood $\widehat{N}$ over which $\boldsymbol{\Theta}^\sharp|_{\widehat{N}}$ is invertible and $\widehat{\boldsymbol{\Theta}}|_{\widehat{N}\cap\{\widehat{\Omega}\}^-} = (\boldsymbol{\Theta}^\sharp|_{\widehat{N}})^{-1}|_{\widehat{N}\cap\{\widehat{\Omega}\}^-}$.

Given any mapping $\boldsymbol{\Phi} \in H^1(\Omega;\mathbb{R}^n)$, the composite mapping $\widehat{\boldsymbol{\Phi}} := \boldsymbol{\Phi} \circ \widehat{\boldsymbol{\Theta}}$ belongs to the space $H^1(\widehat{\Omega};\mathbb{R}^n)$ since the bijection $\boldsymbol{\Theta} : \overline{\Omega} \to \{\widehat{\Omega}\}^-$ is bi-Lipschitzian. Moreover,

$$\widehat{\nabla}\widehat{\boldsymbol{\Phi}}(\hat{x}) = \nabla\boldsymbol{\Phi}(x)\widehat{\nabla}\widehat{\boldsymbol{\Theta}}(\hat{x}) = \nabla\boldsymbol{\Phi}(x)\nabla\boldsymbol{\Theta}(x)^{-1} \quad \text{for almost all } \hat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega},$$

the notation $\widehat{\nabla}$ indicating that differentiation is performed with respect to the variable $\hat{x}$. Hence $\det\widehat{\nabla}\widehat{\boldsymbol{\Phi}} > 0$ a.e. in $\widehat{\Omega}$ if in addition $\det\nabla\boldsymbol{\Phi} > 0$ a.e. in Ω.

By Lemma 2.3, there exists a constant $c_0(\boldsymbol{\Theta}) := \Lambda(\widehat{\Omega})$ with the following property: given any mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfying $\det \nabla\boldsymbol{\Phi} > 0$ a.e. in Ω, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that the mapping $\widehat{\boldsymbol{\Phi}} = \boldsymbol{\Phi} \circ \widehat{\boldsymbol{\Theta}}$ satisfies

$$\|\widehat{\nabla}\widehat{\boldsymbol{\Phi}} - \boldsymbol{R}\|_{L^2(\widehat{\Omega};\mathbb{M}^n)} \leqslant c_0(\boldsymbol{\Theta}) \big\|\widehat{\nabla}\widehat{\boldsymbol{\Phi}}^{\mathrm{T}}\widehat{\nabla}\widehat{\boldsymbol{\Phi}} - \boldsymbol{I}\big\|^{1/2}_{L^1(\widehat{\Omega};\mathbb{S}^n)}.$$

The injectivity of the mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ and the relation $\det \nabla\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$ together imply that

$$\begin{aligned}
\|\widehat{\nabla}\widehat{\boldsymbol{\Phi}} - \boldsymbol{R}\|^2_{L^2(\widehat{\Omega};\mathbb{M}^n)} &= \int_{\widehat{\Omega}} \big|\widehat{\nabla}\widehat{\boldsymbol{\Phi}}(\hat{x}) - \boldsymbol{R}\big|^2 \,\mathrm{d}\hat{x} = \int_{\Omega} \big|\nabla\boldsymbol{\Phi}(x)\nabla\boldsymbol{\Theta}(x)^{-1} - \boldsymbol{R}\big|^2 \det\nabla\boldsymbol{\Theta}(x)\,\mathrm{d}x \\
&\geqslant \int_{\Omega} \big|\nabla\boldsymbol{\Phi}(x) - \boldsymbol{R}\nabla\boldsymbol{\Theta}(x)\big|^2 \big|\nabla\boldsymbol{\Theta}(x)\big|^{-2} \det\nabla\boldsymbol{\Theta}(x)\,\mathrm{d}x \\
&\geqslant c_1(\boldsymbol{\Theta}) \|\nabla\boldsymbol{\Phi} - \boldsymbol{R}\nabla\boldsymbol{\Theta}\|^2_{L^2(\Omega;\mathbb{M}^n)},
\end{aligned}$$

where $c_1(\boldsymbol{\Theta}) := \inf_{x\in\overline{\Omega}}\{|\nabla\boldsymbol{\Theta}(x)|^{-2}\det\nabla\boldsymbol{\Theta}(x)\} > 0$. Likewise,

$$\begin{aligned}
\big\|\widehat{\nabla}\widehat{\boldsymbol{\Phi}}^{\mathrm{T}}\widehat{\nabla}\widehat{\boldsymbol{\Phi}} - \boldsymbol{I}\big\|_{L^1(\widehat{\Omega};\mathbb{S}^n)} &= \int_{\widehat{\Omega}} \big|\widehat{\nabla}\widehat{\boldsymbol{\Phi}}(\hat{x})^{\mathrm{T}}\widehat{\nabla}\widehat{\boldsymbol{\Phi}}(\hat{x}) - \boldsymbol{I}\big|\,\mathrm{d}\hat{x} \\
&= \int_{\Omega} \big|\nabla\boldsymbol{\Theta}(x)^{-T}\big(\nabla\boldsymbol{\Phi}(x)^{\mathrm{T}}\nabla\boldsymbol{\Phi}(x) - \nabla\boldsymbol{\Theta}(x)^{\mathrm{T}}\nabla\boldsymbol{\Theta}(x)\big)\nabla\boldsymbol{\Theta}(x)^{-1}\big| \det\nabla\boldsymbol{\Theta}(x)\,\mathrm{d}x \\
&\leqslant c_2(\boldsymbol{\Theta}) \big\|\nabla\boldsymbol{\Phi}^{\mathrm{T}}\nabla\boldsymbol{\Phi} - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\big\|_{L^1(\Omega;\mathbb{S}^n)},
\end{aligned}$$

where $c_2(\boldsymbol{\Theta}) := \sup_{x\in\overline{\Omega}}\{|\nabla\boldsymbol{\Theta}(x)^{-\mathrm{T}}||\nabla\boldsymbol{\Theta}(x)^{-1}|\det\nabla\boldsymbol{\Theta}(x)\} < \infty$. The announced inequality thus holds with $c(\boldsymbol{\Theta}) := c_0(\boldsymbol{\Theta})c_1(\boldsymbol{\Theta})^{-1/2}c_2(\boldsymbol{\Theta})^{1/2}$. $\square$

Lemma 2.5. *Let the assumptions on the set Ω and the mapping $\boldsymbol{\Theta}$ be as in Lemma* 2.4. *Then there exists a constant $C(\boldsymbol{\Theta})$ with the following property*: *given any mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfying* $\det\nabla\boldsymbol{\Phi} > 0$ *a.e. in Ω, there exist a vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ and a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$\big\|\boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R}\boldsymbol{\Theta})\big\|_{H^1(\Omega;\mathbb{R}^n)} \leqslant C(\boldsymbol{\Theta}) \big\|\nabla\boldsymbol{\Phi}^{\mathrm{T}}\nabla\boldsymbol{\Phi} - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\big\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Proof. Let there be given any mapping $\boldsymbol{\Phi} \in H^1(\Omega; \mathbb{R}^n)$ satisfying $\det\nabla\boldsymbol{\Phi} > 0$ a.e. in Ω. By Lemma 2.4, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that

$$\|\nabla\boldsymbol{\Phi} - \boldsymbol{R}\nabla\boldsymbol{\Theta}\|_{L^2(\Omega;\mathbb{M}^n)} \leqslant c(\boldsymbol{\Theta}) \big\|\nabla\boldsymbol{\Phi}^{\mathrm{T}}\nabla\boldsymbol{\Phi} - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\big\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Let the vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ bedefined by

$$\boldsymbol{b} := \bigg(\int_{\Omega} \mathrm{d}x\bigg)^{-1} \int_{\Omega} \big(\boldsymbol{\Phi}(x) - \boldsymbol{R}\boldsymbol{\Theta}(x)\big)\,\mathrm{d}x.$$

By the generalized Poincaré inequality, there exists a constant d such that, for all $\boldsymbol{\Psi} \in H^1(\Omega; \mathbb{R}^n)$,

$$\|\boldsymbol{\Psi}\|_{H^1(\Omega;\mathbb{R}^n)} \leqslant d\bigg(\|\nabla\boldsymbol{\Psi}\|_{L^2(\Omega;\mathbb{M}^n)} + \bigg|\int_{\Omega} \boldsymbol{\Psi}(x)\,\mathrm{d}x\bigg|\bigg).$$

Applying this inequality to the mapping $\boldsymbol{\Psi} := \boldsymbol{\Phi} - (\boldsymbol{b} + \boldsymbol{R}\boldsymbol{\Theta})$ yields the desired conclusion, with $C(\boldsymbol{\Theta}) := dc(\boldsymbol{\Theta})$. $\square$

3. Commentary

A mapping $\boldsymbol{\Theta} \in H^1(\Omega; \mathbb{R}^n)$, resp. $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$, is *orientation-preserving* if $\det \nabla\boldsymbol{\Theta} > 0$ a.e. in Ω, resp. $\det \nabla\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$. Two orientation-preserving mappings $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$ and $\boldsymbol{\Theta} \in H^1(\Omega; \mathbb{R}^n)$ are *isometrically equivalent* if there exist a vector $\boldsymbol{b}$ in $\mathbb{R}^n$ and a rotation $\boldsymbol{R} \in \mathbb{O}^n_+$ such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{b} + \boldsymbol{R}\boldsymbol{\Theta}(x) \quad \text{for almost all } x \in \Omega.$$

Clearly, two such isometrically equivalent mappings share the same Cauchy–Green tensor field in $L^1(\Omega; \mathbb{S}^n)$.

One application of the key inequality of Theorem 2.1 is the following *sequential continuity property*: let $\boldsymbol{\Theta}^k \in H^1(\Omega; \mathbb{R}^n)$, $k \geqslant 1$, and $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ be orientation-preserving mappings. Then there exist a constant $C(\boldsymbol{\Theta})$ and orientation-preserving mappings $\widetilde{\boldsymbol{\Theta}}^k \in H^1(\Omega; \mathbb{R}^n)$, $k \geqslant 1$, that are isometrically equivalent to $\boldsymbol{\Theta}^k$ such that

$$\big\|\widetilde{\boldsymbol{\Theta}}^k - \boldsymbol{\Theta}\big\|_{H^1(\Omega;\mathbb{R}^n)} \leqslant C(\boldsymbol{\Theta})\big\|\big(\nabla\boldsymbol{\Theta}^k\big)^{\mathrm{T}}\nabla\boldsymbol{\Theta}^k - \nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\big\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Hence *the sequence* $(\widetilde{\boldsymbol{\Theta}}^k)_{k=1}^\infty$ *converges to* $\boldsymbol{\Theta}$ *in* $H^1(\Omega; \mathbb{R}^n)$ *as* $k \to \infty$ *if the sequence* $((\nabla\boldsymbol{\Theta}^k)^{\mathrm{T}}\nabla\boldsymbol{\Theta}^k)_{k=1}^\infty$ *converges to* $\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}$ *in* $L^1(\Omega; \mathbb{S}^n)$ *as* $k \to \infty$.

In nonlinear three-dimensional elasticity, such a sequential continuity could thus prove to be useful when considering *infimizing sequences* of the total energy, in particular for handling the part of the energy that takes into account the applied forces and the boundary conditions, which are both naturally expressed in terms of the deformation itself.

Indeed, an alternative approach to the existence theory of Ball [3] could conceivably regard *the Cauchy–Green tensor as the primary unknown*, instead of the deformation itself as is usually the case. This observation, already made by Antman [2], is one of the reasons underlying the present study, the other being differential geometry *per se*. As such, it is a continuation of the works initiated in Ciarlet and Laurent [5] and Ciarlet and Mardare [6]. Note that a similar study, this time motivated by *nonlinear shell theory* and accordingly carried out for *surfaces in* $\mathbb{R}^3$ has been also undertaken in Ciarlet [4] and then extended in Ciarlet and Mardare [7].

More precisely, the continuity of (equivalence classes of isometrically equivalent) mappings in the space $\mathcal{C}^3(\Omega; \mathbb{R}^n)$ as functions of their Cauchy–Green tensor in the space $\mathcal{C}^2(\Omega; \mathbb{S}^n)$, both spaces being equipped with their standard Fréchet topologies, has been established in Ciarlet and Laurent [5]. Note that, in the same spirit but by means of a different approach, the local Lipschitz-continuity of (equivalence classes of isometrically equivalent) mappings in the Banach space $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ as functions of their Cauchy–Green tensor in the Banach space $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)$ has been recently established by Ciarlet and Mardare [6].

Such results are to be compared with the earlier, pioneering estimates of John [11,12] and Kohn [13], which implied *continuity at rigid body deformations*, i.e., at a mapping $\boldsymbol{\Theta}$ that is isometrically equivalent to the identity mapping of $\overline{\Omega}$. The recent and noteworthy result of Reshetnyak [15] for *quasi-isometric mappings* is similar to the one obtained here (as it also deals with Sobolev type norms) and is thus particularly relevant to the present study.

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