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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 477–480
COMPTES RENDUS MATHEMATIQUE

Dynamical Systems

The horocycle flow without minimal sets ☆

M. Kulikov

Theory of Dynamical Systems, Mathematics Department, Moscow State University, Moscow, 119992, Russia

Received 15 September 2003; accepted after revision 29 December 2003

Presented by Étienne Ghys

Abstract

We construct an example of a Fuchsian group such that the corresponding horocycle flow has no minimal sets. ***To cite this article: M. Kulikov, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Le flot horicyclique sans ensemble minimal. On construit un exemple de groupe Fuchsien pour lequel le flot horocyclique est sans ensemble minimal. ***Pour citer cet article : M. Kulikov, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction.

One considers the group $G = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$ as a group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}^2 = \{x + \mathrm{i}y \in \mathbb{C} \mid y > 0\}$ with the metric $\mathrm{d}l^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$. If Γ is a Fuchsian group (= discrete subgroup of G), one considers on $\Gamma\backslash G$ (which is isomorphic to the unit tangent bundle to the surface $\mathbb{H}^2/\Gamma$ of constant negative curvature) the (contracting) *horocycle flow* $u_{\mathbb{R}}$ given by the right action of the one-parameter subgroup $\{u_t\colon\ t \in \mathbb{R}\}$, where $u_t = \left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$. If Γ is a uniform lattice then $u_{\mathbb{R}}$ is minimal and if Γ is a non-uniform lattice then the only $u_{\mathbb{R}}$-minimal sets are periodic orbits (e.g., see [4]). The case of infinitely generated Γ is not well studied yet, and here we construct the following example:

Theorem 1.1. *There exists Fuchsian group Γ such that the horocycle flow on $\Gamma\backslash G$ has no minimal sets.*

This seems to be the first example of such a flow of algebraic nature (smooth flows without minimal sets were constructed in [2,5]). Note that any homogeneous flow on space of finite volume always has a minimal set [8], while in our example $\mathrm{vol}(\Gamma\backslash G) = \infty$.

We skip several technical details here (see detailed exposition in [6]).

☆ This work is supported by grant No. NSh-457.2003.1 of The Ministry of Industry and Science of Russia.
E-mail address: kulikov@mccme.ru (M. Kulikov).

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doi:10.1016/j.crma.2003.12.027

2. Idea of proof

We use here some facts about the limit set and the classification of its points which one can find in [3,7]. The *limit set* $\Lambda = \Lambda(\Gamma) \subset \partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ for a Fuchsian group Γ consists of all accumulation points of the orbit Γz for some (hence, any) $z \in \mathbb{H}^2$. Let $\pi : G \to \Gamma\backslash G$ be the projection and $\mathrm{Vis}_+ : \mathrm{T}^1\mathbb{H}^2 \to \partial\mathbb{H}^2$ be the visual map. The non-wandering set $\Omega_+ \subset \Gamma\backslash G$ of $u_\mathbb{R}$ equals to $\pi(\mathrm{Vis}_+^{-1}(\Lambda))$. There is a disjoint decomposition $\Lambda = \Lambda_h \cup \Lambda_p \cup \Lambda_d \cup \Lambda_{irr}$, where the sets of *horocycle* points Λ_h, *parabolic* points Λ_p, *discrete* points Λ_d and *irregular* points Λ_{irr} consist of limit points such that corresponding horocycles $vu_\mathbb{R}$, $v \in \pi(\mathrm{Vis}_+^{-1}(\xi))$, are, respectively, dense in Ω_+, periodic, closed nonperiodic, and neither dense nor closed in Ω_+. There is a simple geometrical description of these classes (e.g., see [7]). For instance, $\xi \in \Lambda_h$ iff for some (hence, any) $z \in \mathbb{H}$ and any $w \in \mathbb{H}$ there exists $\gamma \in \Gamma$ such that $\gamma(z) \in \mathrm{Int}(O_\xi(w))$, where $O_\xi(z) \subset \mathbb{H}^2$ is the horocycle based at $\xi \in \partial\mathbb{H}^2$ through $z \in \mathbb{H}^2$ (= a Euclidean circle or a line through z tangent to $\mathbb{R}$ at ξ) and $\mathrm{Int}(O_\xi(z))$ is its interior.

Here we introduce a new class Λ_s of limit points with *the shift property*: a point $\xi \in \Lambda_s$ iff $\xi \in \Lambda$ and for some (hence, any) $v \in \pi(\mathrm{Vis}_+^{-1}(\xi))$, there exists a real $t \neq 0$ such that $\overline{vu_\mathbb{R}} \cap \overline{vu_\mathbb{R} g_t} \neq \emptyset$, where $g_\mathbb{R}$ is the geodesic flow on $\Gamma\backslash G$. Since $v \in \pi(\mathrm{Vis}_+^{-1}(\Lambda_h))$ implies $\overline{vu_\mathbb{R}} = \Omega_+$, we have $\Lambda_h \subset \Lambda_s$.

Lemma 2.1. *If $\Lambda = \Lambda_s$ and $\Lambda_{irr} \neq \emptyset$ then Ω_+ does not contain $u_\mathbb{R}$-minimal subsets.*

Idea of proof. Assume there exists $u_\mathbb{R}$-minimal set $C \subset \Omega_+$. Condition $\Lambda_{irr} \neq \emptyset$ implies $C \neq \Omega_+$. Fixing a point in $\mathbb{H}^2$, by means of the Busemann cocycle the set of all horocycles in $\mathrm{T}^1\mathbb{H}^2$ can be identified with $\partial\mathbb{H}^2 \times \mathbb{R}$, and Γ action on it is a skew product over Γ-action on $\partial\mathbb{H}^2$ with translations of $\mathbb{R}$ in fibers. The set $C' = \pi^{-1}(C)/u_\mathbb{R} \subset \Lambda \times \mathbb{R}$ is Γ-minimal, which together with $\Lambda = \Lambda_s$ implies $C' \cap (\{\xi\} \times \mathbb{R}) \supset (\{\xi\} \times \{k + q\mathbb{Z}\})$ for some $\xi \in \Lambda,\ k, q \in \mathbb{R},\ q \neq 0$. This fact and minimality of Γ-action on Λ (e.g., see [1]) implies $C' = \Lambda \times \mathbb{R}$, hence $C = \Omega_+$. Contradiction. □

By *semicircle* here we always mean a Euclidean semicircle $S \subset \mathbb{H}^2$ with diameter contained in $\mathbb{R}$. Denote its center $c(S) \in \mathbb{R}$ and Euclidean radius $r(S)$. For semicircles S and S' with $r(S) = r(S'),\ |c(S) - c(S')| \geqslant 2r(S)$, denote $h(S, S')(z) = c(S) + c(S') - \overline{\mathrm{inv}_S(z)}$, where inv_S is the inversion relative to S. Then $h(S, S') \in G$ and $h(S, S')(\mathrm{Ext}(S)) = \mathrm{Int}(S')$, where Ext and Int are the exterior and the interior of a semicircle, respectively. If $I(h)$ is the *isometric* circle of $h \in G$ then $I(h(S, S')) = S$.

Let us say that a family of semicircles $\{S_l\}_{l\in\mathbb{Z}\setminus\{0\}}$ forms a *crocodile* with coefficient $K \in (0, 1)$ on a segment $[a, b] \subset \mathbb{R}$ iff $r(S_{\pm 2l}) = r(S_{\pm(2l-1)}) = K^{l-1}(1 - K)(b - a)/8,\ l \in \mathbb{N}$, and the diameters form a sequence of 'commutators' $S_1, S_2, S_{-1}, S_{-2}, \ldots, S_{2l-1}, S_{2l}, S_{-2l+1}, S_{-2l}, \ldots$ (see Fig. 1).

Lemma 2.2. *Suppose a Fuchsian group has a system of generators containing $h(S_l, S_{-l}),\ l \in \mathbb{N}$, where a family of semicircles $\{S_l\}_{l\in\mathbb{Z}\setminus\{0\}}$ forms a crocodile on a segment $[a, b]$. Then $b \in \Lambda_s$.*

Proof is a direct calculation.

Let $Q,\ K$ and κ be such that $Q > 1,\ \kappa \in (0, 1),\ K \in (0, 1)$ and $(1 + \kappa)/(1 - \kappa) < Q$. For any $k \in \mathbb{N}$, consider (see Fig. 2) semicircles $S_{\pm(2k-1),0}$ with $r(S_{\pm(2k-1),0}) = \kappa Q^k,\ c(S_{\pm(2k-1),0}) = \pm Q^k$, and put for any $k \in \mathbb{Z}$, $h_{2k-1,0} = h(S_{2k-1,0}, S_{-2k+1,0})$. For any $k \in \mathbb{Z}$, consider also a family of semicircles $\{S_{2k,l}\}_{l\in\mathbb{Z}\setminus\{0\}}$ which forms a crocodile on $[c(S_{2k-1,0}) + r(S_{2k-1,0}), c(S_{2k+1,0}) - r(S_{2k+1,0}))$ with coefficient K; for any $k \in \mathbb{Z},\ l \in \mathbb{Z} \setminus \{0\}$,

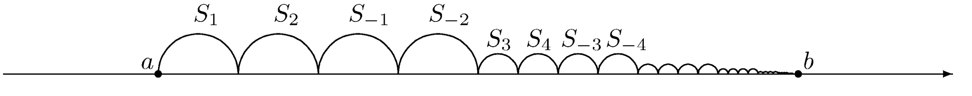

Fig. 1. Crocodile.

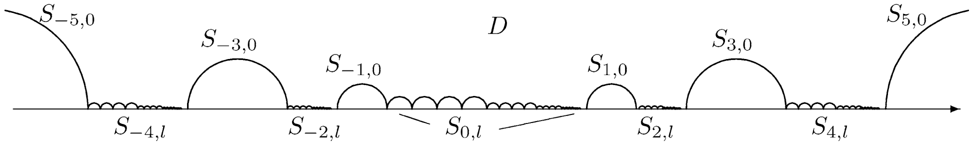

Fig. 2. Generators of Γ.

put $h_{2k,l} = h(S_{2k,l}, S_{2k,-l})$. Let D denote the exterior to all $S_{k,l}$. Consider the Fuchsian group $\Gamma = \langle H\rangle$, $H = \{h_{k,l}\colon k \in 2\mathbb{Z}+1,\ l = 0 \text{ or } k \in 2\mathbb{Z},\ l \in \mathbb{N}\}$.

Let us analyze $\Lambda_h(\Gamma)$. Our main tool (Proposition 2.5 below) bases on the following simple observation:

Lemma 2.3 (See [3, Proof of Theorem 5.4]). *Let $\xi \neq \infty$. Suppose there exist $\{\gamma_n\} \subset \Gamma$ and $\{w_n\} \subset \{z \in \mathbb{H}^2\colon \operatorname{Re} z = \xi\}$ such that $w_n \to \xi$, $n \to \infty$, and $\gamma_n(w_n) \notin \operatorname{Int}(O_{\gamma_n(\xi)}(i))$. Then $\xi \in \Lambda_h$.*

Let d be the Euclidean distance on half-plane $\mathbb{H}^2$, and for a semicircle S, put $J(M,S) = (d(M,c(S)) - r(S))/\max\{r(S), |c(S)|, 1\}$.

Lemma 2.4. *For any $a > 0$, there exists $\delta = \delta(a) > 0$ such that for any two semicircles S and S' with $J(S',S) > a$ and for any $\xi \in \overline{\operatorname{Int}(S)} \cap \mathbb{R}$, the following holds:* $\operatorname{Int}(O_\xi(i)) \cap \operatorname{Int}(S') \subset \{z\colon \operatorname{Im} z \geqslant \delta\}$.

Proof relays on easy computations (see [6]).

If D is a fundamental domain for Γ then for any $\xi \in \Lambda \setminus \Gamma(\infty)$, one can define (not uniquely, in general) a *geometric code*, that is is a sequence $(h^{(j)})_{j=1}^{\infty}$ such that $h^{(j)} \in H \cup H^{-1}$, $h^{(j)} \neq (h^{(j+1)})^{-1}$, $j \in \mathbb{N}$, and $h^{(1)} \cdots h^{(n)}(z_0) \to \xi$ for some $z_0 \in \mathbb{H}^2$ (e.g., see [3]). Denote $h_-^{(j)} = (h^{(j)})^{-1}$, $S^{(j)} = I(h^{(j)})$ and $S_-^{(j)} = I(h_-^{(j)})$ One says that the geometric code contains *simple (complex) jump* of length $a > 0$ at position j iff $\mathrm{JS}(j) > a$ (respectively, $\mathrm{JC}(j) > a$), where $\mathrm{JS}(j) = J(S^{(j-1)}, S_-^{(j)})$ and $\mathrm{JC}(j) = J(h_-^{(j-1)}(S^{(j-2)}), S_-^{(j)})$.

Proposition 2.5. *If a geometric code of a limit point $\xi \in \Lambda \setminus \Gamma(\infty)$ contains infinitely many simple or complex jumps of some fixed length $a > 0$ then $\xi \in \Lambda_h$.*

Idea of proof. Consider here only the case of simple jumps. For any n, choose $w_n \in \overline{h^{(1)} \cdots h^{(n)}(D)} \cap \{z \in \mathbb{H}^2\colon \operatorname{Re} z = \xi\}$ (then $w_n \to \xi$). Given $l_n \geqslant n+2$ define $\gamma_n = h_-^{(l_n-1)} \cdots h_-^{(1)}$. Then $\gamma_n(\xi) \in \overline{\operatorname{Int}(S_-^{(l_n)})}$ and $\gamma_n(w_n) \in h_-^{(l_n-1)} \cdots h_-^{(n+1)}(D) \subset G_{l_n-1,n+1} \subset \overline{\operatorname{Int}(S^{(l_n-1)})}$, where $G_{r,s} = h_-^{(r)} \cdots h_-^{(s+1)}(\overline{\operatorname{Int}(S^{(s)})}) \subset \overline{\operatorname{Int}(S^{(r)})}$, $r \geqslant s$ (see Fig. 3). Now one can show that if we take l_n large enough, then we have $G_{l_n-1,n+1} \subset \{z\colon \operatorname{Im} z < \delta\}$. If we require in addition for l_n, that the inequality $\mathrm{JS}(l_n) > a$ holds, then $G_{l_n-1,n+1} \cap \operatorname{Int}(O_{\gamma_n(\xi)}(i)) = \emptyset$ by Lemma 2.4, hence $\gamma_n(w_n) \notin \operatorname{Int}(O_{\gamma_n(\xi)}(i))$. Because n has been chosen arbitrary, Lemma 2.3 says $\xi \in \Lambda_h$, and the proof is over.

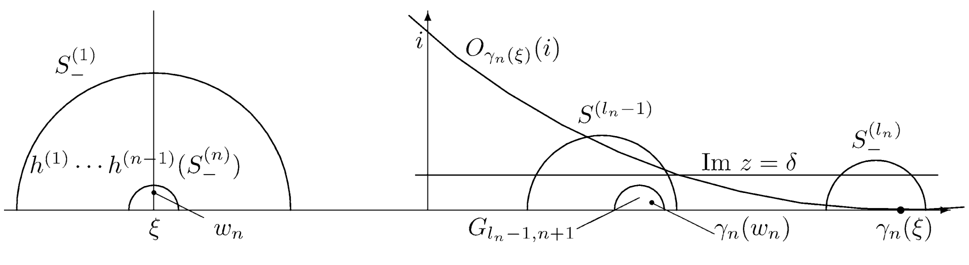

Fig. 3. Proof of Proposition 2.5.

One can show that $Q,\ \kappa,\ K$ can be chosen such that for some $C > 0$ we have (a) $|k_1 - k_2| > 1 \Rightarrow J(S_{k_1,l_1}, S_{k_2,l_2}) > C$, (b) $|k| > 2m+1 \Rightarrow J(\{z\colon |\operatorname{Re} z| \leqslant c(S_{2m+1,0})\}, S_{k,l}) > C$, (c) $|k| < 2m+1 \Rightarrow J(\{z\colon |\operatorname{Re} z| \geqslant c(S_{2m+1,0})\}, S_{k,l}) > C$ and (d) D is a fundamental domain for Γ. While the choice of constants satisfying (a), (b) and (c) bases on easy computations, condition (d) is not so simple (see [6]).

Lemma 2.6. *If for any $a > 0$ a geometric code $(h^{(j)}),\ h^{(j)} = h_{k_j,l_j}$, of a limit point $\xi \in \Lambda \setminus \Gamma(\infty)$ contains finitely many complex jumps of length a then there exist $j_0, m \in \mathbb{N}$ such that for all $j \geqslant j_0$ we have $2m-3 \leqslant |k_j| \leqslant 2m-1$.*

Idea of proof. One can show using conditions (a), (b) and (c) above, that every time if some $2m-1,\ m \in \mathbb{N}$, is contained between $|k_j|$ and $|k_{j+1}|$, then the geometric code contains a simple or a complex jump of length C at position j (see details in [6]).

Proposition 2.7. *For $Q,\ \kappa$ and K chosen above, we have $\Omega_+ = \Gamma\backslash G,\ \Lambda = \Lambda_s$ and $\Lambda_{irr} \neq \emptyset$.*

Idea of proof. Since diameters of $S_{k,l}$ cover the absolute $\Lambda = \partial\mathbb{H}^2$ and D is a fundamental domain, we have $\Lambda = \partial\mathbb{H}^2$, hence $\Omega_+ = \Gamma\backslash G$. Arguments similar to that of [3] yield $\Gamma(\infty) \subset \Lambda_{irr} \cap \Lambda_s$ (see [6]). It remains to prove $\Lambda \setminus \Gamma(\infty) \subset \Lambda_s$. Since $\Lambda_h \subset \Lambda_s$, Proposition 2.5 and Lemma 2.6 implies that we may restrict ourselves to the case $2m-3 \leqslant |k_j| \leqslant 2m-1,\ j \in \mathbb{N}$, for a geometric code $h^{(j)},\ h^{(j)} = h_{k_j,l_j}$, of a point $\xi \in \Lambda \setminus \Gamma(\infty)$. If $\sup|l_j| < \infty$ then $\xi \in \Lambda(\Gamma_0)$, where $\Gamma_0 = \langle\{h^{(j)}\}\rangle$ is finitely generated, and Lehner's theorem [1] says $\Lambda(\Gamma_0) = \Lambda_h(\Gamma_0) \cup \Lambda_p(\Gamma_0)$. Since $\Lambda_p(\Gamma_0) \subset \Lambda_p(\Gamma) = \emptyset$, $\xi \in \Lambda_h(\Gamma_0) \subset \Lambda_h(\Gamma) \subset \Lambda_s(\Gamma)$.

Assume now $\sup|l_j| = \infty$. Then $|l_{j_s}| \to \infty$ and $\forall s \in \mathbb{N}\ k_{j_s} = 2m$ or $\forall s \in \mathbb{N}\ k_{j_s} = -2m$ for some subsequence $\{j_s\} \subset \mathbb{N}$. Consider only the case $k_{j_s} = 2m$. As in the proof of Proposition 2.5, put $w_n \in h^{(1)} \cdots h^{(n)}(D) \cap \{\operatorname{Re} z = \xi\}$ (then $w_n \to \xi$) and $\gamma(j) = h_-^{(j-1)} \cdots h_-^{(1)}$. Then we get $\gamma(j_s)(\xi) \to \zeta = c(S_{2m+1,0}) - r(S_{2m+1,0}),\ s \to \infty$. Given n, the Euclidean radii of horocycles $R_s^{(n)} = r(\gamma(j_s)(O_\xi(w_n)))$ increase in s, because inversion inv_S increases the radius of any horocycle intersecting semicircle S. If for some $n,\ R_s^{(n)} \to R_0 \in (0,\infty),\ s \to \infty$, then $\gamma(j_s)(O_\xi(w_n)) \to O_\zeta(2R_0),\ s \to \infty$. Hence $\overline{vu_{\mathbb{R}}} \supset wu_{\mathbb{R}}$ for some $v \in \pi(\mathrm{Vis}_+^{-1}(\xi))$, $w \in \pi(\mathrm{Vis}_+^{-1}(\zeta))$. Lemma 2.2 gives $\zeta \in \Lambda_s$, which implies $\xi \in \Lambda_s$.

Otherwise, for any n and some s, $R_s^{(n)} > r(O_{(1+\kappa)Q^m}(i)) \geqslant r(O_{\gamma(j_s)(\xi)}(i))$. This imply $\gamma(j_s)(w_n) \notin \operatorname{Int} O_{\gamma(j_s)(\xi)}(i)$, hence $\xi \in \Lambda_h \subset \Lambda_s$ by Lemma 2.3. We are done.

Finally, Lemma 2.1 and Proposition 2.7 immediately yield Theorem 1.1.

Acknowledgements

The author thanks Professors F. Dal'bo, A.N. Starkov and A.M. Stepin for the statement of problems, help and fruitful discussions and The University-I of Rennes (France) for hospitality during February–March, 2002.

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