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Extension of a Riemannian metric with vanishing curvature

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Abstract

Let Ω be a connected and simply-connected open subset of $\mathbb{R}^n$ such that the geodesic distance in Ω is equivalent to the Euclidean distance. Let there be given a Riemannian metric (g_{ij}) of class $\mathcal{C}^2$ and of vanishing curvature in Ω, such that the functions g_{ij} and their partial derivatives of order $\leqslant 2$ have continuous extensions to $\overline{\Omega}$. Then there exists a connected open subset $\widetilde{\Omega}$ of $\mathbb{R}^n$ containing $\overline{\Omega}$ and a Riemannian metric $(\tilde{g}_{ij})$ of class $\mathcal{C}^2$ and of vanishing curvature in $\widetilde{\Omega}$ that extends the metric (g_{ij}). ***To cite this article: P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Prolongement d'une métrique riemannienne à courbure nulle. Soit Ω un ouvert connexe et simplement connexe de $\mathbb{R}^n$ tel que la distance géodésique dans Ω soit équivalente à la distance euclidienne. Soit (g_{ij}) une métrique riemannienne de classe $\mathcal{C}^2$ et de courbure nulle dans Ω, telle que les fonctions g_{ij} et leurs dérivées partielles d'ordre $\leqslant 2$ aient des extensions continues à $\overline{\Omega}$. Alors il existe un ouvert connexe $\widetilde{\Omega}$ de $\mathbb{R}^n$ contenant $\overline{\Omega}$ et une métrique riemannienne $(\tilde{g}_{ij})$ de classe $\mathcal{C}^2$ et de courbure nulle dans $\widetilde{\Omega}$ qui prolonge la métrique (g_{ij}). ***Pour citer cet article : P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Preliminaries

An integer $n \geqslant 2$ is given once and for all, Latin indices and exponents vary in the set $\{1, 2, \ldots, n\}$, and the summation convention with respect to repeated indices and exponents is used. The notations $\mathbb{S}^n$ and $\mathbb{S}^n_>$ designate the space of all symmetric matrices, and the set of all positive-definite symmetric matrices, of order n. If Ω is an open subset of $\mathbb{R}^n$, we define the set

$$\mathcal{C}^2(\Omega; \mathbb{S}^n_>) := \{\boldsymbol{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^n);\ \boldsymbol{C}(x) \in \mathbb{S}^n_> \text{ for all } x \in \Omega\}.$$

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We define as follows spaces of functions, vector fields, or matrix fields, "*of class* $\mathcal{C}^\ell$ *up to the boundary of* Ω":

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^n$. For any integer $\ell \geqslant 1$, the space $\mathcal{C}^\ell(\overline{\Omega})$ consists of all functions $f \in \mathcal{C}^\ell(\Omega)$ that, together with all their partial derivatives $\partial^{\boldsymbol{\alpha}} f$, $1 \leqslant |\boldsymbol{\alpha}| \leqslant \ell$, can be extended by continuity to $\overline{\Omega}$. Analogous definitions hold for the spaces $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{R}^n)$ and $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{S}^n)$. Any continuous extension to $\overline{\Omega}$ will be identified by a bar.

We also define the set

$$\mathcal{C}^2\big(\overline{\Omega}; \mathbb{S}^n_>\big) := \big\{\boldsymbol{C} \in \mathcal{C}^2\big(\overline{\Omega}; \mathbb{S}^n\big);\ \overline{\boldsymbol{C}}(x) \in \mathbb{S}^n_> \text{ for all } x \in \overline{\Omega}\big\}.$$

Let Ω be a connected open subset of $\mathbb{R}^n$. Given two points $x, y \in \Omega$, a *path joining* x *to* y *in* Ω is any mapping $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^n)$ that satisfies $\boldsymbol{\gamma}(t) \in \Omega$ for all $t \in [0,1]$ and $\boldsymbol{\gamma}(0) = x$ and $\boldsymbol{\gamma}(1) = y$. Given a path $\boldsymbol{\gamma}$ joining x to y in Ω, its *length* is defined by

$$L(\boldsymbol{\gamma}) := \int_0^1 \big|\boldsymbol{\gamma}'(t)\big|\,\mathrm{d}t,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Let Ω be a connected open subset of $\mathbb{R}^n$. The *geodesic distance* between two points $x, y \in \Omega$ is defined by

$$d_\Omega(x, y) = \inf\big\{L(\boldsymbol{\gamma});\ \boldsymbol{\gamma} \text{ is a path joining } x \text{ to } y \text{ in } \Omega\big\}.$$

The following definition is in effect a mild regularity assumption on the boundary of an open subset of $\mathbb{R}^n$:

Definition 1.2. An open subset Ω of $\mathbb{R}^n$ satisfies the *geodesic property* if it is connected and, given any point $x_0 \in \partial\Omega$ and any $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$d_\Omega(x, y) < \varepsilon \quad \text{for all } x, y \in \Omega \cap B(x_0; \delta),$$

where $B(x_0; \delta) := \{y \in \mathbb{R}^n; |y - x| < \delta\}$.

Let a Riemannian metric $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ be given over an open subset Ω of $\mathbb{R}^n$. The Christoffel symbols of the second kind associated with this metric are then defined by

$$\Gamma^k_{ij} := \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}), \quad \text{where } \big(g^{k\ell}\big) := (g_{ij})^{-1},$$

and the mixed components of its associated Riemann curvature tensor are defined by

$$R^p_{\cdot ijk} := \partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik}\Gamma^p_{j\ell} - \Gamma^\ell_{ij}\Gamma^p_{k\ell}.$$

If this tensor vanishes in Ω *and* Ω *is simply-connected*, a classical result in differential geometry asserts that (g_{ij}) is the metric tensor field of a manifold $\boldsymbol{\Theta}(\Omega)$ that is isometrically immersed in $\mathbb{R}^n$. More specifically (see, e.g., Ciarlet and Larsonneur [2, Theorem 2] for an elementary and self-contained proof), *there exists an immersion* $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ *that satisfies*

$$\partial_i \boldsymbol{\Theta}(x) \cdot \partial_j \boldsymbol{\Theta}(x) = g_{ij}(x) \quad \text{for all } x \in \Omega,$$

and, if in addition Ω *is connected, such an immersion is unique up to isometries in* $\mathbb{R}^n$.

In [3] (see [4] for a complete proof), we indicated how a *manifold with boundary*, i.e., a subset of $\mathbb{R}^n$ of the form $\boldsymbol{\Theta}(\overline{\Omega})$, can be likewise recovered from a metric tensor field that, together with its partial derivatives of order $\leqslant 2$, can be continuously extended to the *closure* $\overline{\Omega}$ in such a way that the continuous extensions of the matrices (g_{ij})

remain positive-definite in $\overline{\Omega}$. More specifically, in [3,4] the above existence and uniqueness result is extended as follows "up to the boundary of Ω":

Theorem 1.3. *Let Ω be a simply-connected open subset of $\mathbb{R}^n$ that satisfies the geodesic property* (see Definition 1.2). *Let there be given a matrix field $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$* (in the sense of Definition 1.1) *that satisfies*

$$R^p_{\cdot ijk} = 0 \quad \text{in } \Omega.$$

Then there exists a mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ (again in the sense of Definition 1.1) *that satisfies* (the notations $\overline{\partial_i \boldsymbol{\Theta}}$ and $\overline{g_{ij}}$ represent the continuous extensions of the fields $\partial_i \boldsymbol{\Theta}$ and of the functions g_{ij}, according to Definition 1.1):

$$\overline{\partial_i \boldsymbol{\Theta}}(x) \cdot \overline{\partial_j \boldsymbol{\Theta}}(x) = \overline{g_{ij}}(x) \quad \text{for all } x \in \overline{\Omega},$$

and such a mapping is unique up to isometries in $\mathbb{R}^n$.

2. Another definition of the space $\mathcal{C}^\ell(\overline{\Omega})$

The final *objective of this Note* is to provide sufficient conditions guaranteeing that a Riemannian metric $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ with a Riemann curvature tensor vanishing in an open subset Ω of $\mathbb{R}^n$ can be extended to a Riemannian metric $(\tilde{g}_{ij}) \in \mathcal{C}^2(\widetilde{\Omega}; \mathbb{S}^n_>)$ on a connected open set $\widetilde{\Omega}$ containing $\overline{\Omega}$, in such a way that the Riemann curvature tensor associated with this extension still vanishes in $\widetilde{\Omega}$ (see Theorem 3.1).

To this end, another characterization of the space $\mathcal{C}^\ell(\overline{\Omega})$ is needed (see Theorem 2.2). This is why we first introduce another notion of "geodesic property", stronger than that introduced in Definition 1.2.

Definition 2.1. An open subset Ω of $\mathbb{R}^n$ satisfies the *strong geodesic property* if it is connected and there exists a constant C_Ω such that

$$d_\Omega(x, y) \leqslant C_\Omega |x - y| \quad \text{for all } x, y \in \Omega,$$

where d_Ω designates the geodesic distance in Ω (cf. Section 1).

Remarks. (1) Any connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary satisfies the strong geodesic property; for a proof, see, e.g., Proposition 5.1 in Anicic, Le Dret and Raoult [1].

(2) The strong geodesic property clearly implies the geodesic property, but not conversely; consider, e.g., a bounded open subset of $\mathbb{R}^2$ whose boundary is a cardioid.

The following theorem, which hinges in particular on a profound result of Whitney [7] shows that, when an open set Ω satisfies the strong geodesic property, the space $\mathcal{C}^\ell(\overline{\Omega})$ introduced in Definition 1.1 admits a remarkably simple characterization. This result will in turn play a key role in the announced extension theorem.

Theorem 2.2. *Let Ω be an open subset of $\mathbb{R}^n$ that satisfies the strong geodesic property. Then for any integer $\ell \geqslant 1$, the space $\mathcal{C}^\ell(\overline{\Omega})$ of Definition* 1.1 *can be also defined as*

$$\mathcal{C}^\ell(\overline{\Omega}) = \{ f|_\Omega \in \mathcal{C}^\ell(\Omega);\ f \in \mathcal{C}^\ell(\mathbb{R}^n) \}.$$

Sketch of proof. The proof, which is only briefly sketched here (see [4] for a complete proof), hinges on the following steps. Note that the assumption that Ω satisfies the strong geodesic property is not needed until part (iv).

(i) To begin with, we list some *notations* used throughout this proof. Given a multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, we let

$$|\boldsymbol{\alpha}| := \sum_i \alpha_i \quad \text{and} \quad \partial^{\boldsymbol{\alpha}} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \quad \text{if } |\boldsymbol{\alpha}| \geqslant 1,$$

$$\mathbf{0} := (0, 0, \ldots, 0) \quad \text{and} \quad \partial^{\mathbf{0}} f := f,$$

$$\mathbf{0}! := 1 \quad \text{and} \quad \boldsymbol{\alpha}! := (\alpha_1!)(\alpha_2!) \cdots (\alpha_n!).$$

If $x = (x_i)$ and $y = (y_i)$ are two points in $\mathbb{R}^n$, we let

$$(y - x)^{\mathbf{0}} := 1 \quad \text{and} \quad (y - x)^{\boldsymbol{\alpha}} := (y_1 - x_1)^{\alpha_1} (y_2 - x_2)^{\alpha_2} \cdots (y_n - x_n)^{\alpha_n}.$$

Concurrently with the multi-index notation $\partial^{\boldsymbol{\alpha}} f$ for partial derivatives we also use the notations

$$\partial_{i_1} f := \frac{\partial f}{\partial x_{i_1}}, \ldots, \partial_{i_1 i_2 \ldots i_m} f := \frac{\partial^m f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}},$$

with the understanding that, whenever a summation involves such indices $i_1, i_2, \ldots, i_m$, then they range in the set $\{1, 2, \ldots, n\}$ independently of each other.

(ii) Let Ω be a connected open subset of $\mathbb{R}^n$, let x and y be two points in Ω, and let $\boldsymbol{\gamma} = (\gamma_i) \in \mathcal{C}^1([0,1]; \mathbb{R}^n)$ be a path joining x to y in Ω. Then any function $f \in \mathcal{C}^m(\Omega)$, $m \geqslant 1$, satisfies the following identity, which may be viewed as a *Taylor formula along a path*:

$$\begin{aligned} f(y) = {} & f(x) + \frac{1}{1!} \partial_{i_1} f(x)(y_{i_1} - x_{i_1}) + \cdots + \frac{1}{(m-1)!} \partial_{i_1 \ldots i_{m-1}} f(x)(y_{i_1} - x_{i_1}) \cdots (y_{i_{m-1}} - x_{i_{m-1}}) \\ & + \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{m-2}} \left(\int_0^{t_{m-1}} \partial_{i_1 \ldots i_m} f\big(\boldsymbol{\gamma}(t_m)\big) \gamma'_{i_m}(t_m) \, \mathrm{d}t_m \right) \gamma'_{i_{m-1}}(t_{m-1}) \, \mathrm{d}t_{m-1} \right) \cdots \mathrm{d}t_2 \right) \gamma'_{i_1}(t_1) \, \mathrm{d}t_1. \end{aligned}$$

(iii) The identity established in (ii) in turn implies the following estimate, which may be viewed as a *generalized mean-valued theorem along a path*: Let Ω be a connected open subset of $\mathbb{R}^n$, let x and y be two points in Ω, let $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^n)$ be a path joining x to y in Ω, and let a function $f \in \mathcal{C}^m(\Omega)$, $m \geqslant 1$, be given. Then

$$\left| f(y) - \sum_{|\boldsymbol{\beta}| \leqslant m} \frac{1}{\boldsymbol{\beta}!} \partial^{\boldsymbol{\beta}} f(x)(y - x)^{\boldsymbol{\beta}} \right| \leqslant L(\boldsymbol{\gamma})^m \left\{ \sum_{|\boldsymbol{\alpha}| = m} \frac{1}{\boldsymbol{\alpha}!} \sup_{z \in \boldsymbol{\gamma}([0,1])} \left| \partial^{\boldsymbol{\alpha}} f(z) - \partial^{\boldsymbol{\alpha}} f(x) \right|^2 \right\}^{1/2},$$

where $L(\boldsymbol{\gamma})$ denotes the length of the path $\boldsymbol{\gamma}$.

(iv) The strong geodesic property then allows to get rid of the dependence on the path $\boldsymbol{\gamma}$ in the estimate found in (iii), according to the following sharpened estimate: Let Ω be an open subset of $\mathbb{R}^n$ that satisfies the strong geodesic property and let a function $f \in \mathcal{C}^m(\overline{\Omega})$, $m \geqslant 1$, be given, the space $\mathcal{C}^m(\overline{\Omega})$ being defined as in Definition 1.1. Then, given any point $x_0 \in \overline{\Omega}$ and any number $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon)$ such that

$$\left| \overline{f}(y) - \sum_{|\boldsymbol{\beta}| \leqslant m} \frac{1}{\boldsymbol{\beta}!} \overline{\partial^{\boldsymbol{\beta}} f}(x)(y - x)^{\boldsymbol{\beta}} \right| \leqslant \varepsilon |y - x|^m \quad \text{for all } x, y \in \overline{\Omega} \cap B(x_0; \delta),$$

where $\overline{f} \in \mathcal{C}^0(\overline{\Omega})$ and $\overline{\partial^{\boldsymbol{\beta}} f} \in \mathcal{C}^0(\overline{\Omega})$, $1 \leqslant |\boldsymbol{\beta}| \leqslant m$, denote the continuous extensions of the functions $f \in \mathcal{C}^0(\Omega)$ and $\partial^{\boldsymbol{\beta}} f \in \mathcal{C}^0(\Omega)$.

(v) Let there be given a function f in the space $\mathcal{C}^\ell(\overline{\Omega})$, $\ell \geqslant 1$, according to Definition 1.1. According to a deep result of Whitney [7], *f is also the restriction to Ω of a function in the space $\mathcal{C}^\ell(\mathbb{R}^n)$ if, for each multi-index $\boldsymbol{\alpha}$ satisfying $0 \leqslant |\boldsymbol{\alpha}| \leqslant \ell$, there exist functions $f_{\boldsymbol{\alpha}} \in \mathcal{C}^0(\overline{\Omega})$ with the following property*: *For any points $x, y \in \overline{\Omega}$ and any multi-index $\boldsymbol{\alpha}$ satisfying $0 \leqslant |\boldsymbol{\alpha}| \leqslant \ell$, let*

$$R_{\boldsymbol{\alpha}}(y; x) := f_{\boldsymbol{\alpha}}(y) - \sum_{|\boldsymbol{\beta}| \leqslant \ell - |\boldsymbol{\alpha}|} \frac{1}{\boldsymbol{\beta}!} f_{\boldsymbol{\alpha} + \boldsymbol{\beta}}(x)(y - x)^{\boldsymbol{\beta}}.$$

Then, given any point $x_0 \in \overline{\Omega}$ *and any number* $\varepsilon > 0$, *there exists* $\delta = \delta(x_0, \varepsilon)$ *such that*

$$\left|R_{\boldsymbol{\alpha}}(y;x)\right| \leqslant \varepsilon|y-x|^{\ell-|\boldsymbol{\alpha}|} \quad \text{for all } x, y \in \overline{\Omega} \cap B(x_0;\delta) \text{ and } 0 \leqslant |\boldsymbol{\alpha}| \leqslant \ell.$$

To verify that this is indeed the case, let $x_0 \in \overline{\Omega}$ and $\varepsilon > 0$ be given. Then the estimate of part (iv) applied to each function $\overline{\partial^{\boldsymbol{\alpha}} f}$, $0 \leqslant |\boldsymbol{\alpha}| \leqslant \ell$, shows that there exists $\delta_{\boldsymbol{\alpha}} = \delta_{\boldsymbol{\alpha}}(x_0, \varepsilon)$ such that

$$\left|\overline{\partial^{\boldsymbol{\alpha}} f} - \sum_{|\boldsymbol{\beta}| \leqslant \ell - |\boldsymbol{\alpha}|} \frac{1}{\boldsymbol{\beta}!} \overline{\partial^{\boldsymbol{\beta}}(\partial^{\boldsymbol{\alpha}} f)}(x)(y-x)^{\boldsymbol{\beta}}\right| \leqslant \varepsilon|y-x|^{\ell-|\boldsymbol{\alpha}|} \quad \text{for all } x, y \in \overline{\Omega} \cap B(x_0;\delta_{\boldsymbol{\alpha}}).$$

Since $\partial^{\boldsymbol{\beta}}(\partial^{\boldsymbol{\alpha}} f)(x) = \partial^{\boldsymbol{\beta}+\boldsymbol{\alpha}} f(x)$ for all $x \in \Omega$, it also follows that $\overline{\partial^{\boldsymbol{\beta}}(\partial^{\boldsymbol{\alpha}} f)}(x) = \overline{\partial^{\boldsymbol{\beta}+\boldsymbol{\alpha}} f}(x)$ for all $x \in \overline{\Omega}$. Therefore Whitney's theorem can be applied, with $f_{\boldsymbol{\alpha}} := \overline{\partial^{\boldsymbol{\alpha}} f}$ and $\delta := \min\{\delta_{\boldsymbol{\alpha}}; 0 \leqslant |\boldsymbol{\alpha}| \leqslant \ell\}$. □

3. Extension of a Riemannian metric with a vanishing curvature

We are now in a position to prove the announced extension result. The notations are the same as in Theorem 1.3.

Theorem 3.1. *Let* Ω *be a simply-connected open subset of* $\mathbb{R}^n$ *that satisfies the strong geodesic property and let there be given a matrix field* $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ *that satisfies*

$$R^p_{\cdot ijk} = 0 \quad \text{in } \Omega.$$

Then there exist a connected open subset $\widetilde{\Omega}$ *of* $\mathbb{R}^n$ *containing* $\overline{\Omega}$ *and a matrix field* $(\tilde{g}_{ij}) \in \mathcal{C}^2(\widetilde{\Omega}; \mathbb{S}^n_>)$ *such that*

$$\tilde{g}_{ij}(x) = g_{ij}(x) \quad \text{for all } x \in \Omega \quad \text{and} \quad \widetilde{R}^p_{\cdot ijk} = 0 \quad \text{in } \widetilde{\Omega},$$

where the functions $\widetilde{R}^p_{\cdot ijk} \in \mathcal{C}^0(\widetilde{\Omega})$ *denote the mixed components of the Riemann curvature tensor associated with the field* $(\tilde{g}_{ij})$.

Proof. Since Ω *a fortiori* satisfies the geodesic property and Ω is simply-connected, there exists by Theorem 1.3 a mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ that satisfies

$$\overline{\partial_i \boldsymbol{\Theta}}(x) \cdot \overline{\partial_j \boldsymbol{\Theta}}(x) = \overline{g_{ij}}(x) \quad \text{for all } x \in \overline{\Omega}.$$

Since Ω satisfies the strong geodesic property, there in turn exists by Theorem 2.2 a mapping $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^3(\mathbb{R}^n; \mathbb{R}^n)$ that satisfies

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{\Theta}(x) \quad \text{for all } x \in \Omega.$$

Let then

$$\tilde{g}_{ij}(x) := \partial_i \widetilde{\boldsymbol{\Theta}}(x) \cdot \partial_j \widetilde{\boldsymbol{\Theta}}(x) \quad \text{for all } x \in \mathbb{R}^n,$$

and define the set

$$U := \left\{x \in \mathbb{R}^n;\ \left(\tilde{g}_{ij}(x)\right) \in \mathbb{S}^n_>\right\},$$

which is open in $\mathbb{R}^n$ and contains $\overline{\Omega}$ (since $\tilde{g}_{ij}(x) = \overline{g_{ij}}(x)$ for all $x \in \overline{\Omega}$). Finally, define the set $\widetilde{\Omega}$ as the connected component of U that contains $\overline{\Omega}$; hence the set $\widetilde{\Omega}$ is open and connected.

Furthermore, the mixed components $\widetilde{R}^p_{\cdot ijk}$ of the Riemann curvature tensor associated with the field $(\tilde{g}_{ij})$ are well defined in the set $\widetilde{\Omega}$ since the matrices $(\tilde{g}_{ij}(x))$ are by construction invertible for all $x \in \widetilde{\Omega} \subset U$.

Because $\tilde{g}_{ij}(x) = \partial_i \widetilde{\boldsymbol{\Theta}}(x) \cdot \partial_j \widetilde{\boldsymbol{\Theta}}(x)$ for all $x \in \widetilde{\Omega}$ and the restriction $\widetilde{\boldsymbol{\Theta}}|_{\widetilde{\Omega}} \in \mathcal{C}^3(\widetilde{\Omega}; \mathbb{R}^n)$ is an immersion, the relations $\widetilde{R}^p_{\cdot ijk} = 0$ in $\widetilde{\Omega}$ are simply the well-known necessary conditions that a Riemannian metric induced by an immersion satisfies. □

A similar *extension theorem for surfaces in* $\mathbb{R}^3$ can be likewise established; see [5,6].

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References

[1] S. Anicic, H. Le Dret, A. Raoult, The infinitesimal rigid displacement lemma in Lipschitz coordinates and application to shells with minimal regularity, in press.
[2] P.G. Ciarlet, F. Larsonneur, On the recovery of a surface with prescribed first and second fundamental forms, J. Math. Pures Appl. 81 (2002) 167–185.
[3] P.G. Ciarlet, C. Mardare, On the recovery of a manifold with boundary in $\mathbb{R}^n$, C. R. Acad. Sci. Paris, Ser. I, in press.
[4] P.G. Ciarlet, C. Mardare, Recovery of a manifold with boundary and its continuity as a function of its metric tensor, in press.
[5] P.G. Ciarlet, C. Mardare, Extension of a surface in $\mathbb{R}^3$, in press.
[6] P.G. Ciarlet, C. Mardare, Recovery of a surface with boundary and its continuity as a function of its two fundamental forms, in press.
[7] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36 (1934) 63–89.