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Numerical Analysis

A mortar element method for hyperbolic problems ☆

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Abstract

A non-conforming finite element method based on non-overlapping domain decomposition is extended to linear hyperbolic problems. The method is based on streamline-diffusion/discontinuous Galerkin methods and the mortar element method. A weak flux continuity condition at the inflow interface is enforced by means of Lagrange multipliers. This weak flux continuity condition replaces the usual mortar condition for elliptic problems, and allows non-matching grids at the subdomain interfaces. ***To cite this article: Y. Bourgault, A. El Boukili, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une méthode de joint pour les problèmes hyperboliques. Une méthode d'éléments finis non conforme basée sur une décomposition de domaine est étendue aux problèmes hyperboliques linéaires. Cette méthode combine les techniques de « streamline diffusion », d'éléments finis discontinus et la méthode de joint. La continuité du flux est imposée faiblement sur la portion entrante de l'interface entre les sous-domaines. Cette condition faible de conservation des flux remplace la condition de joint usuelle pour les problèmes elliptiques, et permet l'usage de maillages non conformes sur les interfaces entre les sous-domaines. ***Pour citer cet article : Y. Bourgault, A. El Boukili, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

La méthode d'éléments de joint permet de résoudre avec une précision optimale des problèmes elliptiques ou paraboliques sur plusieurs sous-domaines avec des grilles non conformes sur les interfaces des sous-domaines. Nous proposons une extension de cette méthode aux équations hyperboliques, utilisant comme problème modèle l'équation de transport linéaire. Avant discrétisation par éléments finis, la méthode de joint s'écrit alors comme : Trouver $u=(u_1,u_2)\in Y$ tel que

$$\sum_{k=1,2}\left\{\int_{\Omega_k}\bigl(\operatorname{div}(\beta_k u_k)+\sigma u_k-f\bigr)v_k\,\mathrm{d}x-\int_{\partial\Omega_k^{\mathrm{in}}\setminus\Gamma}\beta\cdot\mathbf{n}(u_k-g)v_k\,\mathrm{d}s\right\}=0 \tag{1}$$

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pour tout $v=(v_1,v_2)\in Y$, où Y est un espace fonctionnel approprié avec une condition de raccord faible à l'interface basée sur le saut des flux sur les frontières entrantes. Plus précisément, cette condition de raccord s'énonce à l'aide de la forme bilinéaire

$$b(u,\mu)=\int_{\Gamma_1^{\mathrm{in}}}\beta\cdot\mathbf{n}_1(u_1-u_2)\mu\,\mathrm{d}s+\int_{\Gamma_2^{\mathrm{in}}}\beta\cdot\mathbf{n}_2(u_2-u_1)\mu\,\mathrm{d}s. \tag{2}$$

Il est possible de montrer que pour le problème continu, la formulation par sous-domaines de la nouvelle méthode de joint redonne la solution de l'équation de transport sur le domaine entier. On propose ensuite une méthode d'éléments finis en discrétisant chaque sous-domaine de façon indépendante et en imposant la condition de raccord énoncée ci-dessus. La méthode d'éléments finis admet une solution unique stable. Il est possible de reformuler le problème sur un espace d'éléments finis non conforme sur l'interface des sous-domaines en introduisant un multiplicateur de Lagrange, avec un choix d'espace pour le multiplicateur légèrement différent de ce qui est utilisé dans le cas elliptique. En 1-D, des résultats numériques montrent que la méthode de joint corrige l'erreur de consistance des méthodes d'éléments finis discontinus en éliminant complètement le saut sur l'interface entre les sous-domaines. En 2-D et particulièrement dans le cas non conforme, la correction de consistance ne semble pas aussi grande. La méthode d'Uzawa utilisée lors des calculs s'est avérée plus efficace dans le cas hyperbolique qu'elliptique.

1. Introduction

The mortar element method is now very popular to decompose elliptic problems on multiple subdomains [1–5]. The main feature of this method is its ability to deal with nonmatching grids on subdomain interfaces without loosing any accuracy of the global solution, while allowing the parallel computing of the solution. As far as we know, the mortar method has been introduced for elliptic or parabolic PDEs only. Its extension to hyperbolic problems (such as the Euler equations for inviscid flows) or mixed-type equations (such as the Navier–Stokes equations for compressible flows) would be a definite asset.

The present paper is an initial step into the development of an "all-at-once" mortar methods that works for all type of equations, first concentrating on its development for the linear advection equation. The proposed mortar method works for hyperbolic equations, through a combination of streamline-diffusion upwinding, discontinuous and mortar finite element terms in the Galerkin formulation. A weak flux continuity condition at the subdomain interface is enforced by means of Lagrange multipliers which yields a solution with optimal accuracy even with non-matching grids at subdomain interfaces. The new method is first introduced in the continuous case in the next section and then in the discrete case in Section 3. Numerical results follow in Section 4.

2. Weak multi-domain formulation with mortar method

As a simple hyperbolic problem, we concentrate on the steady linear advection equation:

$$\begin{aligned}&\operatorname{div}(\beta u)+\sigma u=f\quad \text{in } \Omega,\\ &u=g\quad \text{on } \partial\Omega^{\mathrm{in}},\end{aligned} \tag{3}$$

where β stands for the advective velocity, Ω is a domain with a piecewise twice differentiable boundary, so that the unit normal is defined almost everywhere. The inflow boundary $\partial\Omega^{\mathrm{in}}$ is defined with respect to the outward normal $\mathbf{n}$.

A variational formulation of problem (3) is given by: find $u \in X$ such that

$$\int_{\Omega} \big(\operatorname{div}(\beta u) + \sigma u - f\big) v \, \mathrm{d}x - \int_{\partial\Omega^{\mathrm{in}}} \beta \cdot \mathbf{n}(u - g) v \, \mathrm{d}s = 0 \tag{4}$$

for all $v \in X$. The variational formulation makes sense and well-posedness of the problem can be shown [6] in

$$X = \big\{ v \in L^2(\Omega) \mid \operatorname{div}(\beta v) \in L^2(\Omega),\ v \in L^2_\beta\big(\partial\Omega^{\mathrm{in}} \cup \partial\Omega^{\mathrm{out}}\big)\big\} \tag{5}$$

with L^2_β denoting the usual weighted L^2 space with weight $|\beta \cdot \mathbf{n}|$, provided that $\beta \in W^{1,\infty}(\Omega)$ and $\sigma + \frac{1}{2}\operatorname{div}(\beta) \geqslant \bar{\sigma} > 0$.

We next restate the same problem using a partition of the 2-dimensional domain Ω into two non-overlapping subdomains Ω_1 and Ω_2 with a twice differentiable interface denoted by Γ. The interface Γ could be divided into two parts, $\Gamma_1^{\mathrm{in}} = \{x \in \Gamma \mid \beta(x) \cdot \mathbf{n}(x) < 0\}$ and $\Gamma_2^{\mathrm{in}} = \{x \in \Gamma \mid \beta(x) \cdot \mathbf{n}(x) > 0\}$, using the unit normal vector $\mathbf{n}$ on Γ directed from Ω_1 to Ω_2. Γ_1^{in} and Γ_2^{in} are inflow boundaries for Ω_1 and Ω_2 respectively. We denote by u_k and β_k the restriction of u and β to Ω_k, $k = 1, 2$, respectively. For the domain decomposition formulation, the solution $u = (u_1, u_2)$ and test functions will be taken in $\overline{X} = \prod_{k=1,2} X(\Omega_k)$, where $X(\Omega_k)$ is the restriction to Ω_k of functions in X.

The interface conditions require that the normal flux $\beta \cdot \mathbf{n}u$ is continuous across Γ, namely $\beta_1 \cdot \mathbf{n}u_1 = \beta_2 \cdot \mathbf{n}u_2$ on Γ. Using the mortar space $M = L^2_\beta(\Gamma_1^{\mathrm{in}} \cup \Gamma_2^{\mathrm{in}})$ and the bilinear form $b : \overline{X} \times M \to \mathbb{R}$ defined by

$$b(u, \mu) = \int_{\Gamma_1^{\mathrm{in}}} \beta \cdot \mathbf{n}_1(u_1 - u_2)\mu \, \mathrm{d}s + \int_{\Gamma_2^{\mathrm{in}}} \beta \cdot \mathbf{n}_2(u_2 - u_1)\mu \, \mathrm{d}s, \quad \forall \mu \in M, \tag{6}$$

this amounts to look for a solution in the subspace

$$Y = \big\{ v \in \overline{X} \mid b(v, \mu) = 0,\ \forall \mu \in M \big\}. \tag{7}$$

The problem (4) thus states as: find $u = (u_1, u_2) \in Y$ such that

$$\sum_{k=1,2} \left\{ \int_{\Omega_k} \big(\operatorname{div}(\beta_k u_k) + \sigma u_k - f\big) v_k \, \mathrm{d}x - \int_{\partial\Omega_k^{\mathrm{in}} \setminus \Gamma} \beta \cdot \mathbf{n}(u_k - g) v_k \, \mathrm{d}s \right\} = 0 \tag{8}$$

for all $v = (v_1, v_2) \in Y$. It can be shown that the subspace Y is nothing but the space X where existence and uniqueness of u is guaranteed. Using the terminology of mixed and hybrid formulation [7], the space Y is nothing but the kernel of the operator $B : \overline{X} \to M$ induced by the bilinear form $b(\cdot\,,\cdot)$. By introducing a Lagrange multiplier $\lambda \in M$, the problem (8) could be rewritten as a mixed formulation on the whole space $\overline{X} \times M$ and the inf-sup condition can be shown for the bilinear form $b(\cdot\,,\cdot)$. More precisely,

Theorem 2.1. $\operatorname{Ker}(B)$ *is isomorphic to X and if all streamlines of the velocity field β intersect the inflow boundary $\partial\Omega^{\mathrm{in}}$, then, for any $\lambda \in M$, there exists $u_\lambda \in \overline{X}$ such that $Bu_\lambda = \lambda$ and $\|u_\lambda\|_{\overline{X}} \leqslant C\|\lambda\|_M$ for some constant C independent from λ.*

We recall that the streamline going through a point $x \in \Omega$ is the curve $y = y(\tau; x)$ parameterized by τ, solution of the initial value problem $y'(\tau) = \beta(y(\tau; x))$ and $y(\tau_0) = x$.

3. Finite element discretization

For each $1 \leqslant k \leqslant 2$, we associate a family of triangular finite element meshes T_{kh} of Ω_k, and we denote by X_{kh} the space of piecewise linear finite element on T_{kh}. Let h_k be the maximal diameter of the elements of T_{kh}

and $\overline{X}_h = \prod_{k=1,2} X_{kh}$ the discrete product space. Note that the meshes do not need to match at the interfaces. In order to build a finite element space approaching Y, one has to write a weak flux continuity constraint at the inflow interfaces. Let us define a space of mortar functions or Lagrange multipliers on the interface. We denote by Tr_k the trace operator from $X(\Omega_k)$ onto $L^2_\beta(\partial\Omega_k)$ and define (among several other possibilities) $\widetilde{M}_{kh} = \{\mathrm{Tr}_k\, v|_{\Gamma_k^{\mathrm{in}}},\ v \in X_{kh}\}$ as the space of piecewise linear polynomials on Γ_k^{in}. The mortar space is given by $M_h = \prod_{k=1,2} \widetilde{M}_{kh}$. Using the bilinear form $b(\cdot\,,\cdot)$ defined above, we are now able to define the subspace Y_h of $\overline{X}_h$:

$$Y_h = \bigl\{v_h \in \overline{X}_h \mid b(v_h, \mu_h) = 0,\ \forall \mu_h \in M_h\bigr\}. \tag{9}$$

Consider a family of $(\delta_k)_{1\leqslant k\leqslant 2}$ of small positive parameters with $\delta_k \sim h_k$, and define the non-symmetric bilinear form $a_{kh} : X_{kh} \times X_{kh} \to \mathbb{R}$:

$$a_{kh}(u_{kh}, v_{kh}) = \int_{\Omega_k} \bigl(\mathrm{div}(\beta u_{kh}) + \sigma u_{kh}\bigr)\bigl(v_{kh} + \delta_k\, \mathrm{div}(\beta v_{kh})\bigr)\, \mathrm{d}x, \tag{10}$$

and the bilinear form $\tilde{a}_h : \overline{X}_h \times \overline{X}_h \to \mathbb{R}$:

$$\tilde{a}_h(u_h, v_h) = \int_{\Gamma_1^{\mathrm{in}}} |\beta\cdot\mathbf{n}|(u_{1h} - u_{2h})v_{1h}\, \mathrm{d}s + \int_{\Gamma_2^{\mathrm{in}}} |\beta\cdot\mathbf{n}|(u_{2h} - u_{1h})v_{2h}\, \mathrm{d}s. \tag{11}$$

Using the bilinear form $a_h : \overline{X}_h \times \overline{X}_h \to \mathbb{R}$ defined by

$$a_h(u_h, v_h) = \sum_{k=1,2} a_{kh}(u_{kh}, v_{kh}) + \tilde{a}_h(u_h, v_h), \tag{12}$$

the discretization of problem (8) states as: find $u_h \in Y_h$ such that

$$a_h(u_h, v_h) = \sum_{k=1,2} \int_{\Omega_k} f\bigl(v_{kh} + \delta_k\, \mathrm{div}(\beta v_{kh})\bigr)\, \mathrm{d}x, \quad \forall v_h \in Y_h. \tag{13}$$

For all $v \in \overline{X}_h$, define the norm

$$|||v|||^2 = \sum_{k=1,2} \Biggl\{ \frac{\bar\sigma}{2} \int_{\Omega_k} v_k^2\, \mathrm{d}x + \frac{\delta_k}{2} \int_{\Omega_k} \bigl(\mathrm{div}(\beta v_k)\bigr)^2\, \mathrm{d}x + \frac{1}{2} \int_{\Gamma_k^{\mathrm{in}}} |\beta\cdot\mathbf{n}_k|(v_2 - v_1)^2\, \mathrm{d}x \Biggr\}. \tag{14}$$

The stability of the method and the well-posedness of the discrete problem (13) can be proven:

Theorem 3.1. *If $\delta_k \leqslant \bar\sigma/(2\sigma^2)$ for all $k \in \{1,2\}$, then, for any $v_h \in Y_h$, $a_h(v_h, v_h) \geqslant |||v_h|||^2$.*

We should notice that by omitting the upwinding terms given by the bilinear form $\tilde{a}_h$, the coerciveness of the bilinear form a_h is lost. The coerciveness of the bilinear form a_h implies the existence and the uniqueness of the solution in Y_h.

The discrete problem can also be posed in the unconstrained spaces $\overline{X}_h$, at the cost of introducing a Lagrange multiplier. It results in the following mixed formulation: find $(u_h, \lambda_h) \in \overline{X}_h \times M_h$ such that

$$\begin{cases} a_h(u_h, v_h) + b(v_h, \lambda_h) = \displaystyle\sum_{k=1,2} \int_{\Omega_k} f(v_{kh} + \delta_k \beta\cdot\nabla v_{kh}), \quad \forall v_h \in \overline{X}_h, \\ b(u_h, \mu_h) = 0, \quad \forall \mu_h \in M_h. \end{cases} \tag{15}$$

The existence of the solution for this problem can be derived using a similar argument as in [3].

4. Numerical results

The method has been tested numerically with two subdomains using a test case with a simple analytical solution for the linear advection equation in one dimension. Fig. 1 shows the spatial evolution of the solution with Uzawa iterations. At convergence, the jump of the solution is null at the mortar interface, meaning that the mortar method reduces (in this 1-D example removes) the usual discretization error associated with discontinuous finite element methods. Fig. 1 (right) illustrates the convergence of the absolute value of the flux jump at the interface with respect to Uzawa iterations, both for advection and advection-diffusion equations. The solution of the mixed formulation with the Uzawa method seems easier for the hyperbolic equation than for the elliptic equation, but a final statement would require more analysis in 2-D. The method has also been tested for the 2-D linear advection equation using two and four subdomains, and compared with the solution on a single domain (see Fig. 2). For this 2-D test case, a square wave is advected by a clockwise rotating velocity field. The solution is continuous along the streamlines but only piecewise continuous over the domain.

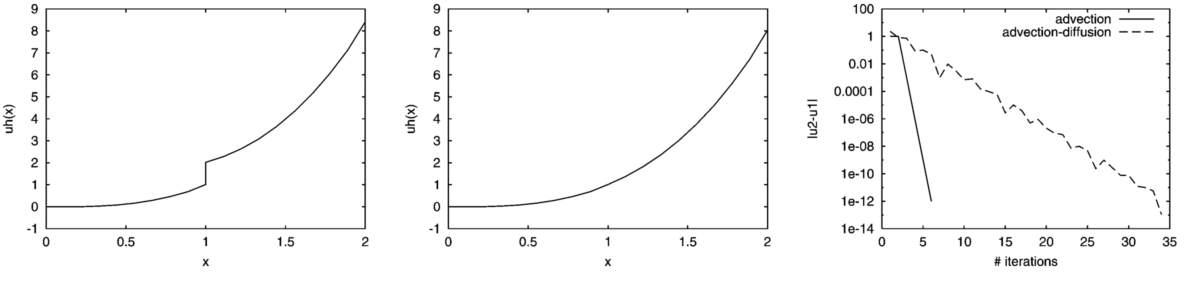

Fig. 1. Numerical solutions after 2 steps (left) and at convergence (middle). Flux jump w/r to Uzawa iterations (right).

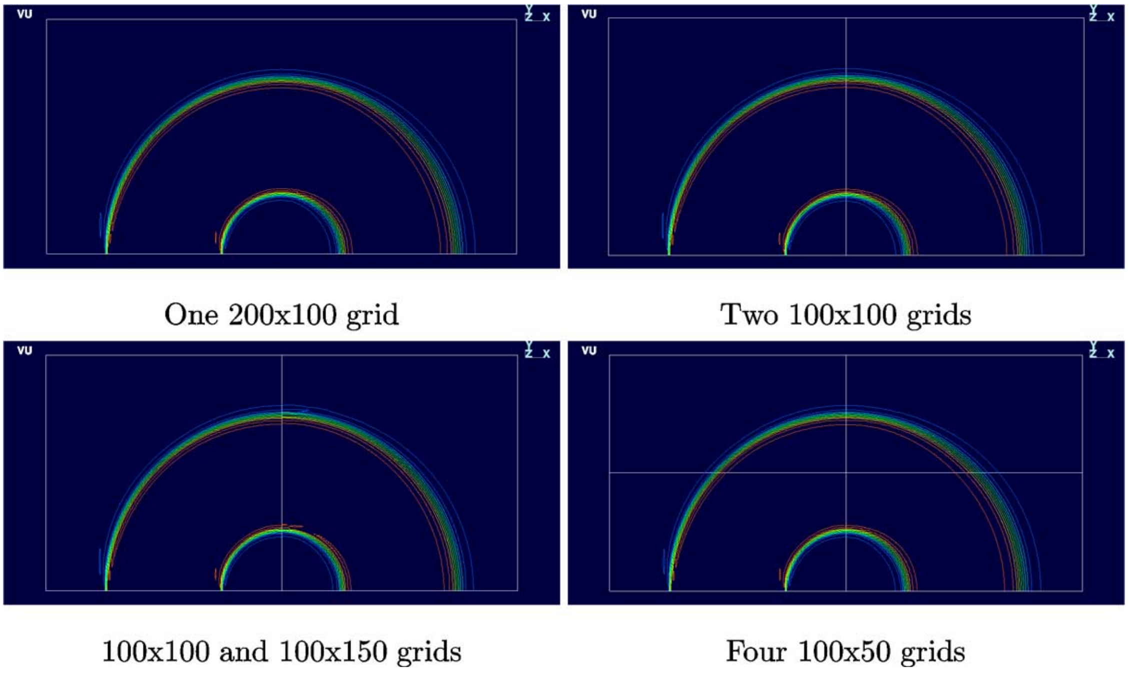

Fig. 2. Pure advection of a scalar in a rotating flow, as obtained with the mortar method over multiple domains using matching and non-matching grids.

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