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Probability Theory

Convergence in variation of the laws of multiple stable integrals

Jean-Christophe Breton

Laboratoire de mathématiques et applications, Université de La Rochelle, avenue Michel Crépeau, 17042 La Rochelle cedex, France

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Abstract

Multiple stable integrals generalize Wiener–Itô integrals, their construction being based upon a generalized LePage representation. This approach allows one to study their behaviour. We are interested in this Note in the continuity for total variation norm of the laws of these integrals $I_d(f)$ with respect to f. ***To cite this article: J.-C. Breton, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Convergence en variation des lois des intégrales stables multiples. Les intégrales stables multiples généralisent celles de Wiener–Itô, leur construction est fondée sur une représentation de LePage généralisée. Cette approche permet d'étudier leur loi. Nous nous intéressons dans cette Note à la continuité pour la variation totale des lois de ces intégrales $I_d(f)$ par rapport à f. ***Pour citer cet article : J.-C. Breton, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Résultat principal

Nous considérons l'intégrale stable multiple $I_d(f)$ par rapport à la mesure α-stable M sur $[0,1]$ (que l'on suppose en plus symétrique si $\alpha \geqslant 1$) définie pour $f \in L^\alpha(\log_+)^{d-1}([0,1]^d)$ dans [1] par et avec la propriété de représentation $I_d(f) \stackrel{\mathcal{L}}{=} S_d(f)$ où $S_d(f) := C_\alpha^{d/\alpha} \sum_{\mathbf{i}>0} [\gamma_\mathbf{i}][\Gamma_\mathbf{i}]^{-1/\alpha} f(\mathbf{V_i})$ est une série de type LePage où les variables aléatoires Γ_i, γ_i, V_i sont données après (3). Ce type d'intégrales généralise celles de Wiener–Itô et vérifie certaines de leurs propriétés telles que l'absolue continuité des lois (cf. [1]). Le résultat principal de cette Note vise à généraliser pour les intégrales stables celui de [3] obtenu dans le cas d'intégrales de Wiener–Itô. Plus précisément, en notant $\xrightarrow{var}$ la convergence en variation, on montre que :

Théorème 0.1. *Soient M comme précédemment (c'est-à-dire vérifiant (2)) et $(f_n)_{n>0}$ convergeant vers $f \not\equiv 0$ dans $L^\alpha(\log_+)^{d-1}([0,1]^d)$, alors on a $\mathcal{L}(I_d(f_n)) \xrightarrow{var} \mathcal{L}(I_d(f))$, $n \to +\infty$.*

Grâce à la régularité des lois prouvée dans [1], on a aussi :

E-mail address: jcbreton@univ-lr.fr (J.-C. Breton).

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Corollaire 0.2. *Avec les mêmes hypothèses que dans le Théorème* 0.1*, on a la convergence dans* $L^1(\mathbb{R})$ *des densités de* $\mathcal{L}(I_d(f_n))$ *vers celle de* $\mathcal{L}(I_d(f))$*.*

La preuve consiste à se ramener à des polynômes aléatoires en conditionnant $S_d(f)$ puis à les étudier grâce aux propriétés des séries de LePage. On commence par quelques rappels sur la méthode de la superstructure utilisée.

Méthode de superstructure

Pour étudier une distribution fonctionnelle Pf^{-1}, Davydov se ramène à des distributions auxiliaires $Q_\varepsilon F_\varepsilon^{-1}$ sur un espace plus large [2] : pour cela, on dispose en général de transformations G_c de l'espace $\mathcal{X}$ avec la propriété $PG_c^{-1} \xrightarrow{var} P$, $c \to 0$, et on considère sur $[0,\varepsilon] \otimes \mathcal{X}$ la mesure $Q_\varepsilon = (1/\varepsilon)\lambda|_{[0,\varepsilon]} \times P$ et la fonctionnelle $F_\varepsilon(c,x) = f(G_c x)$. On a alors $Q_\varepsilon F_\varepsilon^{-1} \xrightarrow{var} Pf^{-1}$, $\varepsilon \to 0$, et $Q_\varepsilon F_\varepsilon^{-1}$ s'exprime comme mélange de mesures en dimension 1 qui sont les mesures images des restrictions de f, comme en (6).

Esquisse de preuve

Soient f_n et f comme dans l'énoncé du le théorème et μ_n, μ les lois de leurs intégrales. On trouve un multi-indice $\mathbf{i}^*$ tel que $f(\mathbf{V}_{\mathbf{i}^*}) \neq 0$ p.s. En supposant que l'espace est un espace produit $(\Omega, \mathcal{F}, P) \otimes (\Omega', \mathcal{F}', P')$ avec $(\Gamma_i)_{i>0}$, et $\mathbf{Y} := (\gamma_i, V_i)_{i>0}$ ne dépendant resp. que de $(\Omega, \mathcal{F}, P)$ et de $(\Omega', \mathcal{F}', P')$, on se ramène d'abord à l'étude de $\mu_{n,y} = \mathcal{L}(S_d(f_n) \mid \mathbf{Y} = y) \stackrel{\mathcal{L}}{=} F_{n,y}(\Gamma^{-1/\alpha})$ où avec $A_{n,\mathbf{i}}(y) = d!C_\alpha^{d/\alpha}[\epsilon_{\mathbf{i}}]f_n(\mathbf{u}_{\mathbf{i}})$ pour $y = (\epsilon, u)$, on a noté $F_{n,y}(x) = \sum_{\mathbf{i} \in T^d} A_{n,\mathbf{i}}(y) x_{i_1} \cdots x_{i_d}$. Notons alors $p(y) := i_d^*$, et pour une suite $t = (t_i)_{i \geqslant 0}$, $t_{\leqslant j} = (t_i)_{i \leqslant j}$, $t_{\geqslant j} = (t_i)_{i \geqslant j}$ les suites tronquées. Alors, en notant

$$\nu := \mathcal{L}\big(\Gamma_1^{-1/\alpha}, \Gamma_2^{-1/\alpha}, \ldots, \Gamma_{p(y)}^{-1/\alpha} \mid y = (\gamma_i, V_i)_i \text{ et } t_{\geqslant p(y)+1} = (\Gamma_i, i \geqslant p(y)+1)\big)$$

qui admet une densité car $\forall n \geqslant 1$, la loi de $(\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$ conditionnellement à $(\Gamma_{n+1}, \Gamma_{n+2}, \ldots)$ en admet une, et en conditionnant maintenant par le futur de Γ, μ et μ_n s'expriment comme mélanges de $\nu(F_{n,y}(\cdot, t_{\geqslant t_{p(y)+1}})^{-1}$ et $\nu(F_y(\cdot, t_{\geqslant t_{p(y)+1}})^{-1}$ respectivement par rapport aux lois de $\mathbf{Y}$ et $(\Gamma_i)_{i \geqslant p(y)+1}$. D'après les propriétés des variations, il suffit dès lors de montrer

$$\mathcal{L}\big(\nu\big(F_{n,y}(\cdot, t_{\geqslant t_{p(y)+1}})\big)^{-1}\big) \xrightarrow{var} \mathcal{L}\big(\nu\big(F_y(\cdot, t_{\geqslant t_{p(y)+1}})\big)^{-1}\big). \tag{1}$$

Observons d'après la définition de $F_{n,y}$, que $R_{n,y,t}(s) = F_{n,y}(s, t_{\geqslant p(y)+1}^{-1/\alpha})$ est un polynôme en $s = (s_1, \ldots, s_{p(y)})$. De même, nous associons un polynôme $R_{y,t}$ à F_y.

Par choix de $\mathbf{i}^*$ et par conditionnement par le futur de Γ, le polynôme $R_{y,t}$ admet comme terme de degré maximal d l'unique monôme $X_{i_1^*} \cdots X_{i_d^*}$. Fixons $v \in (\mathbb{R}_*)^{p(y)}$ de sorte que $c \mapsto R_{y,t}(s + cv)$ est un polynôme non nul de degré maximal d égal à $A_{\mathbf{i}^*}(y) v_{i_1^*} \cdots v_{i_d^*} \neq 0$. Considérons alors la famille de transformations de $\mathbb{R}_+^{p(y)}$ données par $G_c(s) = s + cv$ pour laquelle, par continuité des translations dans $L^1(\mathbb{R}^{p(y)})$, on a $\nu G_c^{-1} \xrightarrow{var} \nu$, $c \to 0$. On applique maintenant la méthode de superstructure sur $(\mathbb{R}^{p(y)}, \nu)$ dont le principe a été décrit ci-dessus. En utilisant les notations correspondantes : par l'analogue de (5), on déduit la convergence uniforme en n quand $\varepsilon \to 0$: $\|\nu R_{n,y,t}^{-1} - \mathcal{Q}_\nu^\varepsilon F_{n,\varepsilon}^{-1}\| \to 0$, ainsi que $\|\nu R_{y,t}^{-1} - \mathcal{Q}_\nu^\varepsilon F_\varepsilon^{-1}\| \to 0$. Mais grâce à (8), on se ramène à l'étude de $\|\mathcal{Q}_\nu^\varepsilon F_\varepsilon^{-1} - \mathcal{Q}_\nu^\varepsilon F_{n,\varepsilon}^{-1}\|$ puis grâce à une écriture du type (6) et aux propriétés de la variation, à $\|\lambda|_{[0,\varepsilon]}(\varphi_{y,t}^{s,v})^{-1} - \lambda|_{[0,\varepsilon]}(\varphi_{n,y,t}^{s,v})^{-1}\| \to 0$, quand $n \to +\infty$ où $\varphi_{n,y,t}^{s,v}$, $\varphi_{y,t}^{s,v}$ sont des polynômes à une indéterminée obtenus à partir de $R_{n,y,t}$ et $R_{y,t}$ en $s + cv$.

Étudions pour cela la convergence des coefficients du polynôme $\varphi_{n,y,t}^{s,v}$ vers ceux de $\varphi_{y,t}^{s,v}$ ce qui permettra d'appliquer la Proposition 3.1 ci-dessous avec $\mathcal{P} = \{\lambda|_{[0,\varepsilon]}\}$ pour obtenir (9). En fait, il suffit d'étudier la convergence des coefficients de $R_{n,y,t}$ vers ceux de $R_{y,t}$ pour $P_{(\Gamma_i)_{i \geqslant p(y)+1}}$-presque chaque $t_{\geqslant p(y)+1}$ et $P'_{\mathbf{Y}}$-presque chaque y. Compte tenu des définitions de $R_{n,y,t}$ et de $F_{n,y}$, on étudie les coefficients aléatoires du type (10), ce qui est possible grâce à l'étude de la continuité en probabilité des séries de LePage dans [1]. Nous obtenons alors la convergence dans le sens précédent des coefficients des polynômes $R_{n,y,t}$ vers ceux de $R_{y,t}$ et donc celle de ceux de $\varphi^{s,v}_{n,y,t}: c \mapsto R_{n,y,t}(s+cv)$.

Finalement par la Proposition 3.1, on a $\|\mathcal{Q}^{\varepsilon}_{\nu} F_{\varepsilon}^{-1} - \mathcal{Q}^{\varepsilon}_{\nu} F_{n,\varepsilon}^{-1}\| \to 0$ quand $n \to +\infty$ puis d'après (8), $\|\nu R_{y,t}^{-1} - \nu R_{n,y,t}^{-1}\| \to 0$, c'est-à-dire (1), ce qui permet finalement de conclure la preuve du théorème.

1. Introduction

We consider on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an α-stable random measure M on $([0,1], \mathcal{B}([0,1]), \lambda)$ with control measure the Lebesgue measure λ and skewness function $\beta:[0,1] \to [-1,1]$ as Samorodnitsky-Taqqu's terminology in [6]. We suppose moreover

$$\text{for } 0<\alpha<1,\ M \text{ is arbitrary}, \quad \text{and} \quad \text{for } 1 \leqslant \alpha < 2,\ M \text{ is symmetric } (\beta \equiv 0). \tag{2}$$

For $f \in L^{\alpha}(\log_+)^{d-1}([0,1]^d) := \{f\colon\ [0,1]^d \to \mathbb{R} \text{ measurable: } \int_{[0,1]^d} |f|^{\alpha}(1+\log_+|f|)^{d-1}\, \mathrm{d}\lambda^d < \infty\}$, we have shown in [1] we can construct multiple stable integral $I_d(f) = \int_{[0,1]^d} f \, \mathrm{d}M^d$ using LePage type series with the representation property $I_d(f) \stackrel{\mathcal{L}}{=} S_d(f)$ where

$$S_d(f) := C_{\alpha}^{d/\alpha} \sum_{\mathbf{i}>0} [\gamma_{\mathbf{i}}][\Gamma_{\mathbf{i}}]^{-1/\alpha} f(\mathbf{V_i}) \tag{3}$$

and C_α is a normalization constant given in [6, Eq. (1.2.9)], $(\Gamma_i)_{i\in\mathbb{N}^*}$ are Gamma random variables independent of the i.i.d. sequence (V_i, γ_i) with V_i uniform on $[0,1]$ and with γ_i given by

$$\mathbb{P}\{\gamma_i = -1 | V_i\} = \frac{1-\beta(V_i)}{2}, \qquad \mathbb{P}\{\gamma_i = +1 | V_i\} = \frac{1+\beta(V_i)}{2}$$

and where for multi-index $\mathbf{i} = (i_1, \dots, i_d)$, $[\gamma_{\mathbf{i}}] = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_d}$, $[\Gamma_{\mathbf{i}}] = \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_d}$. We refer to references in [1,5] for other approaches of multiple stable integrals. Such integrals can be seen as counterparts of Wiener–Itô integrals in the stable context. Several properties of the law of the latest such as absolute continuity can also be derived for the former (see [1]). These integrals are also connected to stable random series (see [4]). In this Note, the goal is to generalize for multiple stable integrals the continuity for variation of the laws of stochastic integrals, obtained in the Wiener–Itô case in [3]. In the sequel $\|\mu\|$ is the total variation of a finite signed measure μ on a space $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ and $\xrightarrow{var}$ stands for the convergence in variation. We state the main result of this Note:

Theorem 1.1. *Let M be as in* (2) *and $(f_n)_{n>0}$ converges to $f \not\equiv 0$ in $L^{\alpha}(\log_+)^{d-1}([0,1]^d)$, then we have*

$$\mathcal{L}\big(I_d(f_n)\big) \xrightarrow{var} \mathcal{L}\big(I_d(f)\big), \quad n \to \infty. \tag{4}$$

From [1], we know $\mathcal{L}(I_d(f_n)) \ll \lambda$, $\mathcal{L}(I_d(f)) \ll \lambda$, therefore due to properties of variation, Theorem 1.1 can also be stated as a local limit theorem:

Corollary 1.2. *With the same hypothesis as in Theorem* 1.1*, the convergence of densities of $\mathcal{L}(I_d(f_n))$ to those of $\mathcal{L}(I_d(f))$ holds in $L^1(\mathbb{R})$.*

The general argument of the proof is to use suitable conditioning in order to apply the superstructure method and to get back to random polynomials that we study using the properties of representation (3). We begin with the description of superstructure and then sketch the proof.

2. Superstructure argument

This method is proposed by Davydov [2, Section 5] to study a functional distribution Pf^{-1} where on a space $\mathcal{X}$, P is a probability measure and f a functional. It consists in introducing auxiliary families of measures Q_ε and of functionals F_ε on a larger space such that $Q_\varepsilon F_\varepsilon^{-1} \xrightarrow{var} Pf^{-1}$ when $\varepsilon \to 0$ and such that $Q_\varepsilon F_\varepsilon^{-1}$ can be expressed as a mixture of measures whose study is more tractable. In a simple case, it works as follows: let be given a family $\{G_c\}_{c\in[0,a]}$ of transformations of $\mathcal{X}$ with $PG_c^{-1} \xrightarrow{var} P$, $c \to 0$. Consider for $\varepsilon \in [0,a]$, on the product space $([0,\varepsilon],\ \mathcal{B}_{[0,\varepsilon]}) \otimes (\mathcal{X},\mathcal{U})$, the following auxiliary families of measures and functionals: $Q_\varepsilon = (1/\varepsilon)\lambda|_{[0,\varepsilon]} \times P$, $F_\varepsilon(c,x) = f(G_c x)$. We derive

$$\|Pf^{-1} - Q_\varepsilon F_\varepsilon^{-1}\| = \left\| \frac{1}{\varepsilon}\int_0^\varepsilon \left(Pf^{-1} - PG_c^{-1}f^{-1}\right)\mathrm{d}c \right\| \leqslant \frac{1}{\varepsilon}\int_0^\varepsilon \|P - PG_c^{-1}\|\,\mathrm{d}c \to 0, \quad \varepsilon \to 0. \tag{5}$$

To study $Q_\varepsilon F_\varepsilon^{-1}$, decompose now the product space into strata parallel to the factor space $[0,\varepsilon]$, and bring back to the study of $\varphi_x(c) = f(G_c x)$, $c \in [0,\varepsilon]$, that is the restrictions of f over orbits of $\{G_c\}_c$:

$$Q_\varepsilon F_\varepsilon^{-1} = \frac{1}{\varepsilon}\int_{\mathcal{X}} \lambda|_{[0,\varepsilon]}\varphi_x^{-1}\,P(\mathrm{d}x). \tag{6}$$

3. Sketch of the proof

Let f_n and f be as in Theorem 1.1. Note $\mu_n = \mathcal{L}(I_d(f_n))$ and $\mu = \mathcal{L}(I_d(f))$. Since it is enough to see (4) for a subsequence, there is no restriction in assuming $(f_n)_{n>0}$ converges to f both in $L^\alpha(\log_+)^{d-1}([0,1]^d)$ and λ^d-a.e. in $[0,1]^d$. We suppose moreover f is symmetric and zero over diagonal terms (see [1]). Note $T^d := \{\mathbf{i} \in \mathbb{N}^d,\ 0 < i_1 < i_2 < \cdots < i_d\}$. We easily show there is $\mathbf{i} \in T^d$ with $\mathbf{V_i} := (V_{i_1}, V_{i_2}, \ldots, V_{i_d}) \in A_f := \{x \in [0,1]^d \mid f(x) \neq 0\}$ with $\lambda^d(A_f) > 0$. Choose $\mathbf{i}^*$ to be the unique of those indices satisfying the following minimal requirements by order of preference $\min i_d, \min i_{d-1}, \ldots, \min i_2, \min i_1$. We have easily $f(\mathbf{V}_{\mathbf{i}^*}) \neq 0$ and $\mathbb{P}\{f_n(\mathbf{V}_{\mathbf{i}^*}) \to f(\mathbf{V}_{\mathbf{i}^*})\} = 1$.

3.1. Conditioning by $(\gamma_i, V_i)_{i>0}$

As $(\gamma_i, V_i)_{i>0}$ and $(\Gamma_i)_{i>0}$ are independent, suppose the basic probability space is a product space $(\Omega, \mathcal{F}, P) \otimes (\Omega', \mathcal{F}', P')$, with $\mathbb{P} = P \otimes P'$ and $(\Gamma_i)_{i>0}$, $(\gamma_i, V_i)_{i>0}$ being defined on each of the factor space, that is $\Gamma_i(\omega,\omega') = \Gamma_i(\omega)$, $\gamma_i(\omega,\omega') = \gamma_i(\omega')$, $V_i(\omega,\omega') = V_i(\omega')$. With $\mathbf{Y} = (\gamma_i, V_i)_{i>0}$ depending only on $(\Omega', \mathcal{F}', P')$, μ_n can be expressed as a mixture of $\mu_{n,y} = \mathcal{L}(S_d(f_n) \mid \mathbf{Y} = y) = PF_{n,y}(\Gamma^{-1/\alpha})^{-1}$ with respect to $P'_{\mathbf{Y}}$, where for $y = (\epsilon, u)$ with $\epsilon \in \{-1,+1\}^{\mathbb{N}}$ and $u \in [0,1]^{\mathbb{N}}$, we note $A_{n,\mathbf{i}}(y) = d!C_\alpha^{d/\alpha}[\epsilon_\mathbf{i}]f_n(\mathbf{u_i})$, and $F_{n,y}(x) = \sum_{\mathbf{i}\in T^d} A_{n,\mathbf{i}}(y)x_{i_1}\cdots x_{i_d}$. The same holds also for μ with an analogous F_y.

3.2. Conditioning by the future of Γ

In the sequel, note $t_{\leqslant j} = (t_i)_{i\leqslant j}$ and $t_{\geqslant j} = (t_i)_{i\geqslant j}$ for the truncated sequences obtained from $t = (t_i)_{i>0}$, note also $p(y) = i_d^*$ and condition with respect to $(\Gamma_i)_{i\geqslant p(y)+1}$. To this way, consider the following measure on $\mathbb{R}^{p(y)}$:

$$\nu := \mathcal{L}\left(\Gamma_1^{-1/\alpha}, \Gamma_2^{-1/\alpha}, \ldots, \Gamma_{p(y)}^{-1/\alpha} \;\middle|\; y = (\gamma_i, V_i)_i \text{ and } t_{\geqslant p(y)+1} = \left(\Gamma_i, i \geqslant p(y)+1\right)\right).$$

It has a density since $\forall n \geqslant 1$, the law of $(\Gamma_1, \ldots, \Gamma_n)$ conditioned by $(\Gamma_{n+1}, \Gamma_{n+2}, \ldots)$ gets one. Measures μ and μ_n can now be expressed as mixtures of resp. $\nu(F_{n,y}(\cdot, t_{\geqslant t_{p(y)+1}}))^{-1}$ and $\nu(F_y(\cdot, t_{\geqslant t_{p(y)+1}}))^{-1}$ with respect to the laws of $\mathbf{Y}$ and of $(\Gamma_i)_{i\geqslant p(y)+1}$. Using elementary properties of variation, it is enough from now on to derive for $P'_{\mathbf{Y}}$-almost all y and $P_{(\Gamma_i)_{i\geqslant p(y)+1}}$-almost all $t_{\geqslant p(y)+1}$ that

$$\mathcal{L}\left(\nu\left(F_{n,y}(\cdot, t_{\geqslant t_{p(y)+1}})\right)^{-1}\right) \xrightarrow{var} \mathcal{L}\left(\nu\left(F_y(\cdot, t_{\geqslant t_{p(y)+1}})\right)^{-1}\right). \tag{7}$$

To this way, we shall apply superstructure method on $(\mathbb{R}^{p(y)}, \nu)$ and functionals $F_{n,y}(\cdot, t_{\geqslant p(y)+1})$ and $F_y(\cdot, t_{\geqslant p(y)+1})$ that are in fact (from the definition of $F_{n,y}$) d-multivariate polynomials. We note them henceforth $R_{n,y,t}(s_1, \ldots, s_{p(y)})$ and $R_{y,t}(s_1, \ldots, s_{p(y)})$.

3.3. Superstructure

Consider $(\mathbb{R}^{p(y)}, \nu)$. From the choice of $\mathbf{i}^*$ and the conditioning by Γ, it appears that $R_{y,t}$ is of degree d with unique monomial of degree d, $X_{i_1^*} \cdots X_{i_d^*}$. For a fixed $v \in (\mathbb{R}_*)^{p(y)}$, $c \in \mathbb{R} \mapsto R_{y,t}(s + cv)$ is a polynomial of degree d with maximal coefficient given by $A_{\mathbf{i}^*}(y) v_{i_1^*} \ldots v_{i_d^*} \neq 0$. Consider the following transformations of $\mathbb{R}_+^{p(y)}$: $G_c(s) = s + cv$. Since ν belongs to $L^1(\mathbb{R}_+^{p(y)})$, convergence in variation of νG_c^{-1} to ν is equivalent to convergence of their densities in $L^1(\mathbb{R}_+^{p(y)})$ which follows from continuity of translation operator G_c, so that $\nu G_c^{-1} \xrightarrow{var} \nu$ holds as $c \to 0$.

Apply then superstructure on $([0,\varepsilon], \mathcal{B}([0,\varepsilon])) \otimes (\mathbb{R}_+^{p(y)}, \mathcal{B}(\mathbb{R}_+^{p(y)}))$ with the following auxiliary families: $\mathcal{Q}_\nu^\varepsilon = (1/\varepsilon)\lambda|_{[0,\varepsilon]} \times \nu$, $F_\varepsilon(c,s) = R_{y,t}(s + cv)$. Expressing $\mathcal{Q}_\nu^\varepsilon F_{n,\varepsilon}^{-1}$ as in (5), we derive as $\varepsilon \to 0$ uniform convergence in n to 0 of $\|\nu R_{n,y,t}^{-1} - \mathcal{Q}_\nu^\varepsilon F_{n,\varepsilon}^{-1}\|$. Similarly $\|\nu R_{y,t}^{-1} - \mathcal{Q}_\nu^\varepsilon F_\varepsilon^{-1}\| \to 0$. But

$$\left\|\nu R_{y,t}^{-1} - \nu R_{n,y,t}^{-1}\right\| \leqslant \left\|\nu R_{y,t}^{-1} - \mathcal{Q}_\nu^\varepsilon F_\varepsilon^{-1}\right\| + \left\|\mathcal{Q}_\nu^\varepsilon F_\varepsilon^{-1} - \mathcal{Q}_\nu^\varepsilon F_{n,\varepsilon}^{-1}\right\| + \left\|\mathcal{Q}_\nu^\varepsilon F_{n,\varepsilon}^{-1} - \nu R_{n,y,t}^{-1}\right\|. \tag{8}$$

It remains therefore to deal with the second summand of the right-hand side of (8) that we first express using $\varphi_{n,y,t}^{s,v}(c) = R_{n,y,t}(s + cv)$ as in (6) in Section 2: $\mathcal{Q}_\nu^\varepsilon F_{n,\varepsilon}^{-1} = \frac{1}{\varepsilon}\int_{\mathbb{R}_+^{p(y)}} \lambda|_{[0,\varepsilon]}(\varphi_{n,y,t}^{s,v})^{-1}\,\nu(\mathrm{d}s)$. From elementary property of variation, we are brought back to study for ν-almost all s:

$$\left\|\lambda|_{[0,\varepsilon]}\left(\varphi_{y,t}^{s,v}\right)^{-1} - \lambda|_{[0,\varepsilon]}\left(\varphi_{n,y,t}^{s,v}\right)^{-1}\right\| \longrightarrow 0, \quad n \to \infty. \tag{9}$$

To this way, we study the convergence of the coefficients of polynomials $\varphi_{n,y,t}^{s,v}$ to their analogous for $\varphi_{y,t}^{s,v}$. The convergence in the $\|\cdot\|_1$-sense (to be defined just below) is therefore easily derived and we apply the following proposition with $\mathcal{P} = \{\lambda|_{[0,\varepsilon]}\}$ so that (9) finally holds.

Proposition 3.1 [2, Theorem 4.5]. *For a bounded interval Δ, let $\mathcal{P}$ be a family of measures defined on $\mathcal{B}(\Delta)$ such that their densities are equicontinuous. Let $f \in C^1(\Delta)$ with $f' \not\equiv 0$ a.e. Then*

$$\lim_{\delta\to 0} \sup_{\mu\in\mathcal{P}} \left\{ \left\|\mu f^{-1} - \mu g^{-1}\right\| \;\middle|\; \|f - g\|_1 := \sup_\Delta |f - g| + \sup_\Delta |f' - g'| \leqslant \delta \right\} = 0.$$

Remark that $\varphi_{y,t}^{s,v}$ is a polynomial with non zero dominant coefficient so that the requirement about non degeneracy of derivative is satisfied and the result applies. Since $\|\lambda|_{[0,\varepsilon]}(\varphi_{n,y,t}^{s,v})^{-1}\| \leqslant \|\lambda|_{[0,\varepsilon]}\| < \infty$, dominated convergence ensures therefore, when $n \to \infty$:

$$\left\|\mathcal{Q}_\nu^\varepsilon F_\varepsilon^{-1} - \mathcal{Q}_\nu^\varepsilon F_{n,\varepsilon}^{-1}\right\| \leqslant \frac{1}{\varepsilon} \int_{\mathbb{R}_+^{p(y)}} \left\|\lambda|_{[0,\varepsilon]}\left(\varphi_{y,t}^{s,v}\right)^{-1} - \lambda|_{[0,\varepsilon]}\left(\varphi_{n,y,t}^{s,v}\right)^{-1}\right\| \nu(\mathrm{d}s) \longrightarrow 0$$

for $P_{(\Gamma_i)_{i\geqslant p(y)+1}}$-almost all $t_{\geqslant p(y)+1}$ and $P'_{\mathbf{Y}}$-almost all y. Finally, we derive from (8), that

$$\lim_{n\to\infty} \left\|\nu R_{y,t}^{-1} - \nu R_{n,y,t}^{-1}\right\| = 0$$

for $P_{(\Gamma_i)_{i\geqslant p(y)+1}}$-almost all $t_{\geqslant p(y)+1}$ and $P'_{\mathbf{Y}}$-almost all y, that is convergence (7), which is enough to conclude by conditioning. It remains therefore to deal with the convergence of coefficients of polynomials $\varphi_{n,y,t}^{s,v}$.

3.4. Study of the coefficients of polynomials $\varphi_{n,y,t}^{s,v}$

Coefficients of $\varphi_{y,t}^{s,v}$ are linear combinations of coefficients of polynomial $R_{y,t}$, the coefficients of the combinations being polynomials in fixed s and v. The same holds true for $\varphi_{n,y,t}^{s,v}$ with identical linear combinations. It is thus sufficient to show the convergence of the coefficients of $R_{n,y,t}$ to those of $R_{y,t}$ for $P_{(\Gamma_i)_{i\geqslant p(y)+1}}$-almost all $t_{\geqslant p(y)+1}$ and $P'_{\mathbf{Y}}$-almost all y. In fact, we study the coefficients of random polynomials

$$d!C_\alpha^{d/\alpha} \sum_{\mathbf{i}|i_1<\cdots<i_k\leqslant i_d^*<i_{k+1}<\cdots<i_d} [\gamma_{\mathbf{i}}] f_n(\mathbf{V_i}) X_{i_1}\cdots X_{i_k} \Gamma_{i_{k+1}}^{-1/\alpha}\cdots\Gamma_{i_d}^{-1/\alpha}.$$

Let $\mathbf{j}\in T^d$ with $j_1<\cdots<j_k\leqslant i_d^*<j_{k+1}$, the coefficient of $X_{j_1}\cdots X_{j_k}$ relative to $\mathbf{j}$ is

$$d!C_\alpha^{d/\alpha} \sum_{\substack{\mathbf{i}|i_d^*<i_{k+1}<\cdots<i_d\\ i_1=j_1,\ldots,i_k=j_k}} [\gamma_{\mathbf{i}}] f_n(\mathbf{V_i})\, \Gamma_{i_{k+1}}^{-1/\alpha}\cdots\Gamma_{i_d}^{-1/\alpha}. \tag{10}$$

We study the convergence of coefficients (10) to their counterparts for f. To this way, we use the study of continuity in probability of S_d derived in [1] (Section 4.1.2 for $0<\alpha<1$ and Section 4.2.3 for $1\leqslant\alpha<2$, $\beta\equiv 0$). We condition on $i_d^*=p$: since $(V_i,\gamma_i)_{i>0}$ are independent, their laws remain unchanged for $i>p$. The study of (10) is thus brought back to those of a $(d-k)$-multiple series of LePage type. Since

- $f_n\to f$ in $L^\alpha(\log_+)^{d-1}([0,1]^d)$ yields convergence in $L^\alpha(\log_+)^{d-k-1}([0,1]^{d-k})$,
- (V_i,γ_i), $i>p$, are independent and independent of $(\Gamma_i)_{i>0}$,

a careful reading of the proof of [1, Sections 4.1.2, 4.2.3] shows that it works even with the restriction $p<i_{k+1}<\cdots<i_d$ over indices. We derive therefore from continuity in probability of S_{d-p} that

$$\sum_{\substack{\mathbf{i}|p<i_{k+1}<\cdots<i_d\\ i_1=j_1,\ldots,i_k=j_k}} [\gamma_{\mathbf{i}}] f_n(\mathbf{V_i}) \Gamma_{i_{k+1}}^{-1/\alpha}\cdots\Gamma_{i_d}^{-1/\alpha} \xrightarrow{\mathbb{P}} \sum_{\substack{\mathbf{i}|p<i_{k+1}<\cdots<i_d\\ i_1=j_1,\ldots,i_k=j_k}} [\gamma_{\mathbf{i}}] f(\mathbf{V_i}) \Gamma_{i_{k+1}}^{-1/\alpha}\cdots\Gamma_{i_d}^{-1/\alpha}.$$

We deduce convergence in probability of (10) to its counterpart and extracting a subsequence, the almost sure convergence of (10). Finally, for $P'_{\mathbf{Y}}$-almost all y, for $P_{(\Gamma_i)_{i\geqslant p(y)+1}}$-almost all $t_{\geqslant p(y)+1}$ the following convergence holds:

$$\sum_{\substack{\mathbf{i}|p<i_{k+1}<\cdots<i_d\\ i_1=j_1,\ldots,i_k=j_k}} A_{n,\mathbf{i}}(y) t_{i_{k+1}}^{-1/\alpha}\cdots t_{i_d}^{-1/\alpha} \longrightarrow \sum_{\substack{\mathbf{i}|p<i_{k+1}<\cdots<i_d\\ i_1=j_1,\ldots,i_k=j_k}} A_{\mathbf{i}}(y) t_{i_{k+1}}^{-1/\alpha}\cdots t_{i_d}^{-1/\alpha}.$$

We obtain convergence in the same sense of the coefficients of $R_{n,y,t}$ to those of $R_{y,t}$ and the same holds true for the coefficients of polynomial $c\mapsto R_{n,y,t}(s+cv)$. This allows us to achieve the proof of Theorem 1.1 as explained previously.

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References

[1] J.-C. Breton, Multiple stable stochastic integrals: series representation and absolute continuity of their law, J. Theoret. Probab. 15 (4) (2002) 877–901.

[2] Y.A. Davydov, M.A. Lifshits, N.V. Smorodina, Local Properties of Distributions of Stochastic Functionals, American Mathematical Society, 1998.

[3] Y.A. Davydov, G.V. Martynova, Limit Behavior of Multiple Stochastic Integral, Nauka, Moscow, 1989, pp. 55–57 (en Russe).

[4] S. Kwapien, W. Woyczynski, Random Series and Stochastic Integrals. Single and Multiple Martingale Methods, second edition, Birkhäuser, 2001.

[5] G. Samorodnitsky, J. Szulga, An asymptotic evaluation of the tail of a multiple symmetric α-stable integral, Ann. Probab. 17 (1989) 1503–1520.

[6] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Random Processes, Chapman and Hall, 1994.