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On the stability and causality of general time-dependent bilinear models

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Abstract

In this Note, sufficient conditions are given for the existence of a causal stable solution for general bilinear time series with time-dependent coefficients. ***To cite this article: A. Bibi, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Sur la stabilité et la causalité des modèles bilinéaires généraux à coefficients dépendant du temps. Dans cette Note, nous donnons des conditions suffisantes d'existence et d'unicité de solution causale et stable pour la classe générale de modèles bilinéaires à coefficients dépendant du temps. ***Pour citer cet article : A. Bibi, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction

Many time series exhibit certain characteristics which cannot be explained using standard autoregressive (*AR*), moving average (*MA*) or mixed autoregressive moving average (*ARMA*) models. Indeed, in many practical situations, the stochastic processes under study are non-stationary and nonlinear (see [2] for further discussion). In this Note, we consider a general bilinear model with time-dependent coefficients. Such models are very attractive and useful tools in describing the behavior of a large class of dynamical time series, for instance, in signal processing, control theory, macroeconomic, financial data, and other fields. Also, it seems to be the simplest extension of the linear model, defined by adding terms to a classical *ARMA* model. Several properties of some bilinear time series with time-dependent coefficients are reviewed recently in [7] and in [2].

Following [7] and [2] a real valued time series $(X_t)_{t\in\mathbb{Z}}$, $\mathbb{Z}=\{0,\pm1,\pm2,\ldots\}$, defined on a probability space $(\Omega,\Im,P)$ is said to be a general bilinear time series with time-dependent coefficients if it admits the representation

$$X_t=\sum_{i=1}^{p}a_i(t)X_{t-i}+\sum_{j=1}^{q}b_j(t)\xi_{t-j}+\sum_{i=1}^{P}\sum_{j=1}^{Q}c_{ij}(t)X_{t-i}\xi_{t-j}+\xi_t \tag{1}$$

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denoted by $TDBL(p,q,P,Q)$ where $(a_i(t))_{1\leqslant i\leqslant p}$, $(b_i(t))_{1\leqslant i\leqslant q}$, and $(c_{ij}(t))_{1\leqslant i\leqslant P,1\leqslant j\leqslant Q}$ are bounded time-dependent coefficients which depend on a finite number of parameters, $(\xi_t)_{t\in\mathbb{Z}}$ is a strong white noise, i.e., a sequence of independent and identically distributed (i.i.d.) random variables. The advantage of the bilinear models approach is that the resulting equations retain a regression structure, so that many algorithms designed for linear models can be applied (see [7]). On the other hand, one major disadvantage in using these models is perhaps that most bilinear models are inherently unstable in the sense that it is possible to find bounded (in probability sense) input $(\xi_t)_{t\in\mathbb{Z}}$ that can derive the output $(X_t)_{t\in\mathbb{Z}}$ to become unbounded. Stability and causality properties are suitable features for stochastic models, since they determine the reliability of the forecasting and control rules as well as consistency of the parameter estimators. However, determining if a time-dependent bilinear model is stable is not a trivial task. As a general definition of stochastic stability we adopt the principle that to input bounded $(\xi_t)_{t\in\mathbb{Z}}$, there must correspond bounded output $(X_t)_{t\in\mathbb{Z}}$. Exact analysis often requires evaluating the largest eigenvalues in absolute values of very large matrices (see [1,2]). This procedure is not useful when the dimension of matrices is large. The objective of this Note is to give sufficient conditions, relatively simple, with respect to these given in [1] and in [2] for the stochastic stability of (1) based on the formal Volterra series representation.

2. Stability and causality

Let B be the lag operator ($BX_t = X_{t-1}$), and define the following time-dependent polynomial operators which correspond to *AR*, *MA* and bilinear part respectively $A_t(B) = 1 - \sum_{i=1}^{p} a_i(t)B^i$, $B_t(B) = \sum_{i=0}^{q} b_i(t)B^i$ with $b_0(t) = 1$ for all t and $C_t(B) = \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij}(t)[B^i, B^j]$ with $[B^i, B^j][X_t, \xi_t] = X_{t-i}\xi_{t-j}$. With these operators, the $TDBL(p,q,P,Q)$ can be represented as

$$A_t(B)X_t = B_t(B)\xi_t + C_t(B)[X_t, \xi_t]. \tag{2}$$

Let $(b_i)_{1\leqslant i\leqslant q}$ and $(c_{ij})_{1\leqslant i\leqslant P,1\leqslant j\leqslant Q}$ be two sequences of positive numbers such that $\sup_{t\in\mathbb{Z}}|b_i(t)| \leqslant b_i$, $\sup_{t\in\mathbb{Z}}|c_{ij}(t)| \leqslant c_{ij}$ and assume the following conditions

Condition 1. There is a positive constant M_ξ such that $|\xi_t| \leqslant M_\xi$ for all $t \in \mathbb{Z}$.

This condition is satisfied when for example the noise process $(\xi_t)_{t\in\mathbb{Z}}$ is i.i.d $\mathcal{N}(0,1)$, M_ξ could be roughly chosen to be 4.

Condition 2. The $A_t(B)$ polynomial is factorable in the sense that there exists functions $(\lambda_i(t))_{1\leqslant i\leqslant p}$ such that $A_t(B) = \prod_{i=1}^{p}(1 - \lambda_i(t)B)$.

Theorem 2.1. *Assume that the bilinear model represented by* (2) *satisfies Conditions* 1 *and* 2. *Then*

$$\begin{cases} \left|\lambda_i(t)\right| \leqslant \lambda_i < 1, \quad i = 1,\dots,p, \ \textit{for all}\ t\in\mathbb{Z}, \\ \beta M_\xi \displaystyle\sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij} < 1 \end{cases}$$

constitute sufficient conditions for the stability and causality of the process $(X_t)_{t\in\mathbb{Z}}$ *generated by* (1). *Furthermore we have* $|X_t| \leqslant (\beta M_\xi \sum_{i=0}^{q} b_i)(1 - \beta M_\xi \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij})^{-1}$ *where* $\beta = \prod_{i=1}^{p}(1-\lambda_i)^{-1}$.

Remark 1. The condition $|\lambda_i(t)| \leqslant \lambda_i < 1$, $i = 1,\dots,p$, for all $t\in\mathbb{Z}$ is equivalently to the roots of characteristic polynomial at time t determined by solving $A_t(z) = 0$ lie outside the unit circle (the traditional causality assumption). This condition is well known from the theory of locally stationary process (see [3]).

Proof. A simple way to develop the formal Volterra series representation is to use the so-called "reversion method" developed by Subba Rao (see [6]) and used by Guégan (see [4]) for general time-invariant bilinear models. We replace ξ_{t-j} by $\lambda\xi_{t-j}$ in (2); we obtain the equation

$$A_t(B)X_t = \lambda B_t(B)\xi_t + \lambda C_t(B)[X_t, \xi_t], \tag{3}$$

where λ is a numerical parameter introduced to facilitate the solution, but ultimately λ is allowed to become unity. We want a solution in the form $X_t = \sum_{i=1}^{\infty} \lambda^i X_i(t)$, we substitute the last series into (3) and then equating powers of λ on both sides we get

$$A_t(B)X_j(t) = \begin{cases} B_t(B)\xi_t, & j = 1, \\ C_t(B)\big[X_{j-1}(t), \xi_t\big], & j \geqslant 2. \end{cases} \tag{4}$$

Solving (4) with the following initialization

$$\xi_t = 0 \quad \text{if } t < 0 \quad \text{and} \quad X_j(t) = 0 \quad \text{if } t < 0, \text{ for } j \geqslant 1, \tag{5}$$

we obtain

$$X_j(t) = \begin{cases} \displaystyle\sum_{s=0}^{t} g(t,s) B_s(B)\xi_s, & j = 1, \\ \displaystyle\sum_{s=0}^{t} g(t,s) C_s(B)\big[X_{j-1}(s), \xi_s\big], & j \geqslant 2, \end{cases} \tag{6}$$

where $g(t,s)$ are the Green functions associated with $A_t(z)$ (see [5]). That is $g(t,s)$ is a function of t in the solution of homogenous difference equation $A_t(z)y_t = 0$ on $\mathbb{Z}$ with initial conditions $g(t,s) = 1$ if $s = t$ and 0 if $s = t+1, \ldots, t+p-1$. By assumption, there is a fixed sequence λ_i such that $|\lambda_i(t)| \leqslant \lambda_i < 1$, $i = 1, \ldots, p$, for all $t \in \mathbb{Z}$. Thus for all $j \geqslant 1$, define the processes

$$X_j^*(t) = \begin{cases} A^{-1}(B)B(B)\xi_t, & j = 1, \\ A^{-1}(B)C(B)\big[X_{j-1}^*(t), \xi_t\big], & j \geqslant 2, \end{cases}$$

where $B(B) = 1 + \sum_{i=1}^{q} b_i B^i$, $C(B) = \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij}[B^i, B^j]$, and $A(B) = \prod_{i=1}^{p}(1 - \lambda_i B)$ with $A^{-1}(B) = \prod_{i=1}^{p}\{\sum_{j=0}^{\infty} \lambda_i^j B^j\}$. Hence, the processes $(X_j(t))_{t\in\mathbb{Z}}$ are bounded by $(X_j^*(t))_{t\in\mathbb{Z}}$. Now if the conditions of the theorem holds, then $|X_1^*(t)| \leqslant \beta M_\xi \sum_{i=0}^{q} b_i$, moreover, we have

$$\big|C(B)\big[X_1^*(t), \xi_t\big]\big| \leqslant \left(\beta M_\xi \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij}\right)\left(\beta M_\xi \sum_{i=0}^{q} b_i\right).$$

Now, it is easy to see that $|X_2^*(t)| \leqslant (\beta M_\xi \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij})(\beta M_\xi \sum_{i=0}^{q} b_i)$. By recurrence, it can be shown that

$$\big|X_k^*(t)\big| \leqslant \left(\beta M_\xi \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij}\right)^{k-1}\left(\beta M_\xi \sum_{i=0}^{q} b_i\right) \quad \text{for } k \geqslant 1. \tag{7}$$

It is straightforward to see that $|X_t| \leqslant \sum_{k\geqslant 1} |X_k^*(t)| \leqslant (\beta M_\xi \sum_{i=0}^{q} b_i)(1 - \beta M_\xi \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij})^{-1}$. This completes the proof.

Remark 2. In fact, $X_t = \sum_{k\geqslant 1} X_k(t)$ is the formal decomposition of the model (1) on the Wiener's chaos in terms of ξ_t and $X_k(t)$ is the kth component of the decomposition. In other hand, from (7) we can see that $|X_t| \leqslant \sum_{k\geqslant 1} M_\xi^k g_k$ where $g_k = (\beta \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij})^{k-1}(\beta \sum_{i=0}^{q} b_i)$. Furthermore, under the conditions of the theorem, if ξ_t is bounded by $(\beta \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij})^{-1}$ we can force X_t to be bounded also.

In the next theorem, we will show that $X_t = \sum_{k=1}^{t+1} X_k(t)$ is the unique stable solution to the time-dependent bilinear model represented by (2).

Theorem 2.2. *Under the hypothesis of Theorem* 2.1 *and with the initialization conditions* (5)*, the sequence* $X_t = \sum_{k=1}^{t+1} X_k(t)$ *is the unique stable solution to the time-dependent bilinear model represented by* (2).

Proof. First, we shows that $X_k(t) = 0$ for $t < k-1$ using induction. This statement is true for $k = 1$. Assume that it holds for $1 \leqslant k \leqslant n$. Therefore, we have $X_n(t) = 0$ for $t < n-1$. Now consider $k = n+1$, by using (4) and the fact that $X_n(t-j) = 0$ for $t-j < n-1$, it follows that $X_{n+1}(t) = 0$ for $t-1 < n-1$, i.e., $t < (n+1)-1$. In the other hand, since $X_{t+1}(t-j) = 0$ for $j \geqslant 1$, it follows immediately that $C_t(B)[X_{t+1}(t), \xi_t] = \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij}(t) X_{t+1}(t-i)\xi_{t-j} = 0$. From (4), we have $A_t(B)\sum_{k=2}^{t+1} X_k(t) = C_t(B)[\sum_{j=2}^{t+1} X_{j-1}(t), \xi_t]$ or equivalently

$$A_t(B)\sum_{k=1}^{t+1} X_k(t) = B_t(B)\xi_t + C_t(B)\left[\sum_{j=1}^{t} X_j(t), \xi_t\right] = B_t(B)\xi_t + C_t(B)\left[\sum_{j=1}^{t+1} X_j(t), \xi_t\right]$$

this show that $\sum_{k=1}^{t+1} X_k(t)$ is a solution to (2). Now, if $X_t^{(1)}$ and $X_t^{(2)}$are two stables solutions to (2) with initialization conditions (5). Then we obtain $A_t(B)Y_t = C_t(B)[Y_t, \xi_t]$, i.e., $Y_t = \sum_{i=1}^{P} a_i(t) Y_{t-i} + \sum_{i=1}^{P}\sum_{j=1}^{Q} c_{ij}(t) Y_{t-i}\xi_{t-j}$ where $Y_t = X_t^{(1)} - X_t^{(2)}$. Since $X_t^{(1)} = X_t^{(2)} = 0$ for $t < 0$, then $Y_t = 0$ for $t < 0$. By induction it is not hard to show that $Y_t = 0$ for $t \geqslant 0$.

Example 1. Consider the *TDBL*(2, 0, 2, 2) model

$$X_t = 1.8\cos(1.5 - \cos\theta t) X_{t-1} - 0.81 X_{t-2} + \sum_{i=1}^{2}\sum_{j=1}^{2} c_{ij}(t) X_{t-i}\xi_{t-j} + \xi_t, \tag{8}$$

where $(\xi_t)_{t\in\mathbb{Z}}$ is a white noise which satisfies Condition 1. The roots of $A_t(z) = 1 - 1.8\cos(1.5 - \cos\theta t)z + 0.81z^2$ are $\frac{1}{0.9}\exp\{\pm \mathrm{i}(1.5 - \cos\theta t)\}$ which lie in the region $|z| > 1$. Then with initialization conditions (5), the model (8) admit a stable solution given by Volterra series expansion (6) if we choose the coefficients $(c_{ij}(t))_{1\leqslant i,j\leqslant 2}$ such that $4\sum_{i=1}^{2}\sum_{j=1}^{2} c_{ij} < (0.1)^2$.

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