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Differential Topology

Braids on surfaces and finite type invariants

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Abstract

We prove that there is no functorial universal finite type invariant for braids in $\Sigma \times I$ if the genus of Σ is positive. ***To cite this article: P. Bellingeri, L. Funar, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Tresses sur les surfaces et invariants de type fini. Nous démontrons qu'il n'y a pas d'invariant universel fonctoriel de type fini pour les tresses dans $\Sigma \times I$, lorsque Σ est une surface orientable de genre positif. ***Pour citer cet article : P. Bellingeri, L. Funar, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

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Soit Σ une surface compacte, connexe et orientable. Le groupe $B(\Sigma, n)$ de tresses à n brins sur Σ est une généralisation naturelle du groupe de tresses d'Artin B_n et du groupe fondamental de Σ. Ces groupes sont apparus pour la première fois dans l'étude des espaces de configurations ([6], voir aussi [4]), bien que certains étaient déjà connus avant (le groupe $B(S^2, n)$ a été introduit par Hurwitz). Des présentations pour ces groupes ont été d'abord trouvées par Scott [17] dans le cas des surfaces fermées. Celles-ci ont été ensuite simplifiées par González-Meneses [8], et améliorées et généralisées pour les surfaces à bord dans [3].

Le $\mathbb{Z}$-module libre engendré par des objets 1-dimensionels plongés (comme les tresses, les entrelacs ou encore les enchevêtrements) admet une filtration naturelle obtenue par la résolution des objets singuliers ayant un nombre fini de points doubles. L'algèbre graduée associée (appelée algèbre de diagrammes) peut être explicitement calculée et elle a des propriétés remarquables de finitude. Dans ce contexte, un invariant universel est une *application* de notre catégorie dans une complétion de l'algèbre de diagrammes, qui induit un isomorphisme au niveau gradué. Le théorème fondamental de la théorie de Vassiliev pour les nœuds (mais aussi pour les enchevêtrements, donc les tresses en particulier) dans $\mathbb{R}^3$ est la construction, due à Kontsevich, d'un invariant universel de type fini à coeffi-

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cients dans $\mathbb{Q}$. Un ingrédient essentiel est l'existence d'un associateur de Drinfel'd à coefficients rationnels. Une propriété remarquable de l'intégrale de Kontsevich est sa *fonctorialité* [14], permettant par exemple de la définir pour les entrelacs en utilisant l'invariant pour les tresses. Dans le cas des tresses classiques on sait construire un invariant universel de type fini à coefficients dans $\mathbb{Z}$ [16], mais on ne sait pas s'il existe un tel invariant qui soit aussi *multiplicatif*.

González-Meneses et Paris [9] ont construit un invariant universel pour les tresses sur les surfaces (fermées) mais qui n'est pas fonctoriel. Le propos de cette Note est de démontrer que en effet leur résultat ne peut pas être amélioré :

Théorème 0.1. *Il n'existe pas d'invariant universel fonctoriel de type fini pour les tresses dans $\Sigma \times I$, lorsque Σ est une surface compacte, orientable de genre $g \geqslant 1$.*

La preuve s'appuie sur la forme explicite des relations dans $B(\Sigma, n)$ et elle ne dépend pas du choix de l'anneau de base considéré.

1. Introduction

Let Σ be a compact, connected and orientable surface. The group $B(\Sigma, n)$ of braids on n strands over Σ is a natural generalization of both the classical braid group B_n and the fundamental group $\pi_1(\Sigma)$. It appeared first in the study of configuration spaces ([6], see also [4]). Presentations for $B(\Sigma, n)$ were derived by Scott [17], further improved by González-Meneses for closed surfaces [8] and finally given a very simple form in [3]. In the case of holed spheres the latter have been previously obtained by Lambropoulou [13].

Let us consider a category of embedded 1-dimensional objects like braids, links, tangles, etc. There is a natural filtration on the free $\mathbb{Z}$-module generated by the objects, coming from the singular objects with a given number of double points. The main feature of this filtration is that the associated grading, which is called the diagrams algebra, can be explicitly computed, and has some salient finiteness properties. By universal finite type invariant one generally means a *map* from our category into some completion of the diagrams algebra, which induces an isomorphism at graded level. For instance the celebrated Kontsevich integral is such a universal invariant. A key ingredient is the existence of a Drinfel'd associator with rational coefficients. Notice that there exists a universal invariant for usual braids over $\mathbb{Z}$ [16], but it is not known whether there exists a multiplicative one over $\mathbb{Z}$. In fact an essential feature of the Kontsevich integral is its *functoriality* [14]: it is precisely this property which enables one to extend the Chen iterated integrals from braids to links [14].

González-Meneses and Paris constructed a universal invariant for braids on surfaces [9], but their invariant is not functorial (i.e., multiplicative). The purpose of this Note is to show that actually their result cannot be improved:

Theorem 1.1. *There does not exist any functorial universal finite type invariant for braids in $\Sigma \times I$, if the surface Σ is of genus $g \geqslant 1$.*

Remark 1. In particular there does not exist any universal invariant for tangles in $\Sigma \times I$ which is functorial with respect to the vertical composition of tangles. Thus the Andersen–Mattes–Reshetikhin invariant [1] cannot be made tangle functorial. Notice that the functoriality is essential if one seeks for an extension to links in arbitrary 3-manifolds. The non-existence holds true a fortiori for the category of tangles in 3-manifolds with boundary, unless these are cylinders over planar surfaces.

2. Preliminaries

2.1. Surface braids. Set $\Sigma_{g,p}$ for the compact orientable surface of genus g with p boundary components. We denote by $\sigma_1, \ldots, \sigma_{n-1}$ the standard generators of the braid group on a disk embedded in $\Sigma_{g,p}$, viewed as elements of $B(\Sigma_{g,p},n)$. Let also $a_1,\dots,a_g,b_1,\dots,b_g,z_1,\dots,z_{p-1}$ be the generators of $\pi_1(\Sigma_{g,p})$, where z_i denotes a loop around the i-th boundary component. Assume that the base point of the fundamental group is the startpoint of the first strand. Then each $\gamma\in\{a_1,\dots,a_g,b_1,\dots,b_g,z_1,\dots,z_{p-1}\}$ can be realized by an element, denoted also by γ, in $B(\Sigma_{g,p},n)$, by considering the braid whose first strand is describing the curve γ and whose other strands are constant. We denote $[a,b]=aba^{-1}b^{-1}$. Following [3] we have:

Theorem 2.1. *A presentation for $B(\Sigma_{g,p},n)$ $(n\geqslant 2)$ is given by*:

(1) *Generators*: $\sigma_1,\dots,\sigma_{n-1},a_1,\dots,a_g,b_1,\dots,b_g,z_1,\dots,z_{p-1}$.

(2.i) *Braid relations*:

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}\quad (i=1,\dots,n-2),\qquad \sigma_i\sigma_j=\sigma_j\sigma_i\quad (\textit{if } |i-j|\geqslant 2).$$

(2.ii) *Commutativity relations*:

$$[a_r,\sigma_i]=[b_r,\sigma_i]=[z_k,\sigma_i]=1\quad (i>1,\ 1\leqslant r\leqslant g,\ 1\leqslant k\leqslant p-1);$$
$$\left[a_r,\sigma_1^{-1}a_r\sigma_1^{-1}\right]=\left[b_r,\sigma_1^{-1}b_r\sigma_1^{-1}\right]=\left[z_k,\sigma_1^{-1}z_k\sigma_1^{-1}\right]=1\quad (1\leqslant r\leqslant g,\ 1\leqslant k\leqslant p-1);$$
$$\left[a_r,\sigma_1^{-1}a_s\sigma_1\right]=\left[a_r,\sigma_1^{-1}b_s\sigma_1\right]=\left[b_r,\sigma_1^{-1}a_s\sigma_1\right]=\left[b_r,\sigma_1^{-1}b_s\sigma_1\right]=\left[z_i,\sigma_1^{-1}z_j\sigma_1\right]=1$$
$$(1\leqslant s<r\leqslant g,\ 1\leqslant j<i\leqslant p-1);$$
$$\left[a_r,\sigma_1^{-1}z_k\sigma_1\right]=\left[b_r,\sigma_1^{-1}z_k\sigma_1\right]=1\quad (1\leqslant r\leqslant g,\ 1\leqslant k\leqslant p-1);$$

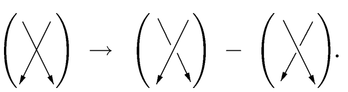
(2.iii) *Skew commutativity relations on each handle* (*when* $g>0$):

$$\left[a_r,\sigma_1^{-1}b_r\sigma_1^{-1}\right]=\sigma_1^2\quad (1\leqslant r\leqslant g);$$

(2.iv) *When* $p=0$ *we have the additional relation*:

$$\left[a_1,b_1^{-1}\right]\left[a_2,b_2^{-1}\right]\cdots\left[a_g,b_g^{-1}\right]=\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1.$$

2.2. The Vassiliev filtration. Singular braids are obtained from braids by allowing a finite number of transverse double points. One can associate to each singular surface braid a linear combination of braids by desingularizing each crossing as follows:



We denote by $\mathcal{V}^d$ the submodule of $\mathbb{Z}[B(\Sigma,n)]$ generated by the desingularizations of (singular) surface braids with d double points. Then $\mathcal{V}^d=J^d$, where $J\subset\mathbb{Z}[B(\Sigma,n)]$ is the two-sided ideal generated by $\{\sigma_i-\sigma_i^{-1}\}_{i=1,\dots,n-1}$. Set $gr^*\mathbb{Z}[B(\Sigma,n)]$ for the associated graded algebra. A morphism of $\mathbb{Z}$-modules $u:\mathbb{Z}[B(\Sigma,n)]\to A$ is a Vassiliev invariant of degree $\leqslant d$ if u vanishes on $\mathcal{V}^{d+1}$.

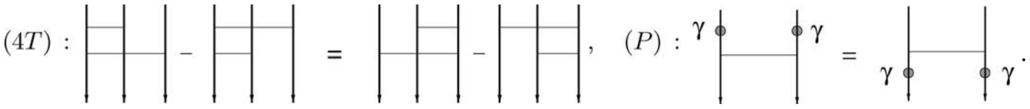
Definition 2.2. A universal Vassiliev invariant for braids is a linear map $Z:\mathbb{C}[B(\Sigma,n)]\to\mathcal{A}$ such that, for any finite type invariant $v:\mathbb{C}[B(\Sigma,n)]\to A$ there exists a homomorphism $u_v:\mathcal{A}\to A$ satisfying $u_v\circ Z=v$. The invariant is functorial if $\mathcal{A}$ has an algebra structure and Z is an algebra homomorphism.

2.3. Presentations of the algebra $gr^*\mathbb{Z}[B(\Sigma,n)]$. Let S_n denote the symmetric group on n elements. The proof given in [9] for $p=0$ extends without essential modifications to the general case:

Proposition 2.1. *The graded algebra $gr^*\mathbb{Z}[B(\Sigma,n)]$ is the quotient of $\mathbb{Z}[Z_{ij}^{\gamma}]_{1\leqslant i,j\leqslant n;\ \gamma\in\pi_1(\Sigma)}\ltimes S_n$, the semi-direct product of a free non-commutative algebra in the given generators with S_n, by the following ideal of relations $\mathcal{R}$*:

$$Z_{ij}^{\gamma}=Z_{ji}^{\gamma^{-1}},\quad \left[Z_{ij}^{\gamma},Z_{kl}^{\delta}\right]=0,\quad \left[Z_{ij}^{\gamma},Z_{jk}^{\delta}+Z_{ik}^{\gamma\delta}\right]=0.$$

This algebra can be described in terms of chord diagrams with beads [7]. Consider the free $\mathbb{Z}$-module generated by the horizontal chord diagrams with endpoints on n vertical segments, where the vertical intervals between two successive chords endpoints are labeled by elements of $\mathbb{Z}[\pi_1(\Sigma)]$ called beads. We obtain a $\mathbb{Z}[\pi_1(\Sigma)]$-algebra $\mathcal{D}(n,\Sigma)$ by imposing the $(4T)$ relations along with (P) below, permitting to push beads across a chord:



Set $F(\Sigma_{g,p}) = \{A_s, B_s, {A_s}^{-1}, {B_s}^{-1}, Z_\alpha, {Z_\alpha}^{-1}\}_{1\leqslant s\leqslant g;\ 1\leqslant\alpha\leqslant p-1}$. The algebra $\mathcal{D}(n,\Sigma)$ is the quotient of the free non-commutative algebra $\mathbb{Z}[Z_{ij}, \gamma^i]_{1\leqslant i,j\leqslant n;\ \gamma\in F(\Sigma_{g,p})}$ by the ideal of relations:

$$[\gamma^i, \delta^j] = 0, \quad [\gamma^i + \gamma^j, Z_{ij}] = 0, \quad \text{for } i \neq j, \text{ and } \gamma, \delta \in F(\Sigma_{g,p}),$$
$$[\gamma^i, Z_{jk}] = 0, \quad Z_{ij} = Z_{ji}, \quad [Z_{ij}, Z_{kl}] = 0, \quad [Z_{ij}, Z_{jk} + Z_{ik}] = 0, \quad \text{if } i,\ j,\ k \text{ are distinct.}$$

If Σ is closed one adds also the relations: $\sum_{s=1}^{g}[A_s^i, B_s^i] = 0$.

Proposition 2.2. *The map $gr^*\mathbb{Z}[B(\Sigma,n)]) \to \mathcal{D}(n,\Sigma) \ltimes S_n$ sending Z_{ij}^{γ} to $\gamma^i Z_{ij} {\gamma^i}^{-1}$, where γ^i denotes the bead labeled γ on the i-th strand, is an isomorphism.*

Remark 2. $\mathcal{D}(n,\Sigma)$ is isomorphic to the algebra of trivalent graphs with beads on oriented edges from [7].

We denote by $H_1(A)$ the abelianization of the graded algebra A, namely the quotient by the homogeneous ideal generated by the elements $ab - ba$, for $a, b \in A$.

Corollary 2.3. $H_1(gr^*\mathbb{Z}[B(\Sigma,n)]) \cong \mathbb{Z}[Z_{12}, A_1^i, {A_1^i}^{-1}, B_1^j, {B_1^j}^{-1}, Z_1^i, {Z_1^i}^{-1}, \tau]/(\tau^2 = 1)$ *as graded commutative algebras. Here τ corresponds to the signature on S_n. The degree of Z_{12} is one, while the other generators are of degree zero.*

3. Proof of the theorem

Assume that such a functorial universal invariant Z exists. The algebra $M = Z(\mathbb{C}[B(\Sigma,n)])$ is filtered by the ideals $\mathcal{V}^j M = Z(\mathcal{V}^j)$ and we set $gr^*(M)$ for the associated graded algebra.

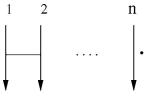
Lemma 3.1. *Z induces an isomorphism of graded algebras $Z^* : gr^*\mathbb{C}[B(\Sigma,n)] \to gr^* M$.*

Proof. It suffices to prove that Z^* is injective. One introduces the pairing $\langle , \rangle : gr^d M \times \mathrm{Hom}(gr^d\mathbb{C}[B(\Sigma,n)], \mathbb{C}) \to \mathbb{C}$, defined as $\langle x, \lambda\rangle = u_{I(\lambda)}(\hat{x})$, where $\hat{x} \in \mathcal{V}^d M$ is a lift of x and $I(\lambda) : \mathbb{C}[B(\Sigma,n)] \to \mathbb{C}$ is an extension of λ as a Vassiliev invariant of degree $\leqslant d$.

Notice that any $\mathbb{Z}$-modules homomorphism $\lambda : \mathcal{V}^d/\mathcal{V}^{d+1} \to A$ can be lifted (non-uniquely) to a Vassiliev invariant $I(\lambda)$ of degree $\leqslant d$, which coincides with λ on $\mathcal{V}^d/\mathcal{V}^{d+1}$.

If $a \in gr^d\mathbb{C}[B(\Sigma,n)] \cap \ker Z^*$, then $\langle Z^*(a), \lambda\rangle = 0$ for all $\lambda \in \mathrm{Hom}(gr^d\mathbb{C}[B(\Sigma,n)], \mathbb{C})$. Choose a representative $\hat{a} \in \mathcal{V}^d$ for a. Then $\langle Z^*(a), \lambda\rangle = u_{I(\lambda)}(Z(\hat{a})) = I(\lambda)(\hat{a}) = \lambda(a)$. This implies $a = 0$ because $gr^d\mathbb{Z}[B(\Sigma,n)]$ is torsion-free (see [9]) and our claim follows. □

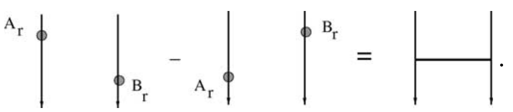
Lemma 3.2. *The map $Z^*_{ab} : H_1(gr^*\mathbb{C}[B(\Sigma,n)]) \to H_1(gr^* M)$ induced by Z^* is trivial in degree one.*

Proof. The image of $\sigma_1^2 - 1$ in $\frac{\mathcal{V}^1}{\mathcal{V}^2}$ is the element Z_{12}, associated to the diagram . Furthermore its image under Z^* is $Z(\sigma_1^2) - 1 \in gr^1(M)$. But $Z(\sigma_1^2) = Z([a_1, \sigma_1^{-1} b_1 \sigma_1^{-1}]) = 1 \in H_1(gr^*(M))$, since Z is multiplicative and the image of a commutator in the abelianization is trivial. Therefore $Z^*_{ab}(Z_{12}) = 0$ and Corollary 2.3 implies our claim. □

The first lemma implies that Z^*_{ab} should be an isomorphism $H_1(gr^*\mathbb{C}[B(\Sigma, n)]) \to H_1(gr^* M)$, while the last lemma shows that this map is trivial in degree one. But the degree one component of $H_1(gr^*\mathbb{C}[B(\Sigma, n)])$ is nonzero from Corollary 2.3, and thus we get a contradiction. This settles the theorem.

4. Comments

Let $PB(\Sigma, n)$ be the pure surface braid group on n strands and $K(\Sigma, n)$ the kernel of the natural homomorphism $\theta : PB(\Sigma, n) \to \pi_1(\Sigma)^n$ which forgets about the braiding and keeps only the fundamental group information of each strand. Then J coincides with the two-sided ideal in $\mathbb{Z}[B(\Sigma, n)]$ generated by the augmentation ideal $I_{K(\Sigma,n)}$ One cannot use the Chen iterated integrals for the subgroup $K(\Sigma, n)$ since this group is not of finite type (see also [5]). An interesting alternative would be to replace the Vassiliev filtration $\mathcal{V}^*$ by that associated to the augmentation ideal $I_{PB(\Sigma,n)}$. The associated graded algebra $gr^*_{I_{PB(\Sigma,n)}}\mathbb{C}[B(\Sigma, n)]$ is now isomorphic to $\mathcal{U}PB(\Sigma, n) \ltimes S_n$, where $\mathcal{U}G$ denotes the universal enveloping algebra of the Lie algebra of the group G over $\mathbb{C}$ (see [11]). In this setting one can find a functorial universal-type invariant (for the new filtration) $\mathbb{C}[B(\Sigma, n)] \to \overline{\mathcal{U}PB(\Sigma, n)} \ltimes S_n$ using the Chen iterated integrals, which takes values in the completion of an algebra of *symplectic chord diagrams*. Specifically $\mathcal{U}PB(\Sigma, n)$ is a chord diagram algebra as defined above, but now the actions of beads on two distinct segments are not anymore assumed to commute, and they are subjected to the following relation:



Nevertheless the grading is different from the previous algebra of chord diagrams since beads generators A_s^i, B_t^j are now given the degree 1 while each chord has degree 2. Alternatively one can present this algebra as the quotient of $\mathbb{C}[A_s^k, B_t^k, Z_{\alpha i}, Z_{ij}]_{1\leqslant s,t\leqslant g,\ 1\leqslant\alpha\leqslant p,\ 1\leqslant i,j,k\leqslant n}$, where $\deg A_s^k = \deg B_t^k = 1$ and $\deg Z_{mj} = 2$ (unless the case when $g = 0$ and one can renormalize the degree of the generators A_{ij} to be 1) by the following relations:

- The extended infinitesimal braid relations:

$$Z_{ij} = Z_{ji}, \quad [Z_{ij}, Z_{kl}] = 0, \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset, \quad [Z_{ij}, Z_{jk} + Z_{ik}] = 0, \quad \text{if } i,\ j,\ k \text{ are distinct.}$$
$$[Z_{\alpha j}, Z_{kl}] = 0,\ \text{if } j \notin \{k, l\}, \quad [Z_{\alpha j}, Z_{\beta k}] = 0,\ \text{if } \{\alpha, j\} \cap \{\beta, k\} = \emptyset, \quad [Z_{\alpha j}, Z_{\alpha k} + Z_{jk}] = 0,\ \text{if } j \neq k.$$

- The relations coming from the fundamental group of Σ:

$$\big[A_s^i, A_r^k\big] = \big[B_s^i, B_r^k\big] = 0, \quad \text{if } i \neq k, \quad \big[A_s^i, B_r^j\big] = 0, \quad \text{if } r \neq s \text{ and } i \neq j,$$
$$\sum_{s=1}^{g} \big[A_s^k, B_s^k\big] + \sum_{j=1}^{n} Z_{jk} + \sum_{\alpha=n+1}^{n+p} Z_{\alpha k} = 0.$$

- The mixed relations:

$$\big[Z_{jk}, A_s^i\big] = 0,\ \text{if } i \notin \{j, k\}, \quad \big[Z_{\alpha k}, A_s^i\big] = 0,\ \text{if } i \neq k, \quad \big[A_s^j + A_s^k, Z_{jk}\big] = 0, \quad \big[B_s^j + B_s^k, Z_{jk}\big] = 0.$$

- The twist relation (making sense only for $g \geqslant 1$):

$$[A_s^i, B_s^j] = Z_{ij}, \quad \text{if } i \neq j.$$

The closed case is proved in [2], for the general case see ([11], Theorem 12.6). A different approach based on the weight filtration on $PB(\Sigma, n)$ is given in [15]. It is not difficult to see that the completion of this symplectic chord diagrams algebra surjects onto $\overline{\mathcal{U}K(\Sigma, n)} \otimes \overline{\mathcal{U}(\pi_1(\Sigma))}$ and hence it furnishes an invariant in the usual chord diagrams but whose coefficients are now formal series from $\overline{\mathcal{U}(\pi_1(\Sigma))}$ instead of merely elements of $\pi_1(\Sigma)$. The relation of the latter with the Vassiliev filtration seems quite obscure.

It is worth mentioning that the universal flat Chen connection whose monodromy yields this invariant is *not quadratic* since the respective configuration spaces are not formal unless $g = 0$ or $n = 2$ (see [2]).

Remark 3. In the case of surface pure braid groups the exact sequence $1 \to \pi_1(\Sigma \setminus \{(n-1) \text{ points}\}) \to PB(\Sigma, n) \to PB(\Sigma, n-1) \to 1$ is not split unless Σ is a torus or Σ has non-empty boundary (see [10]). Moreover even when split, the groups $PB(\Sigma, n)$ are not almost-direct products if $g \geqslant 1$, $n \geqslant 3$. This explains why the arguments in [12] fail for higher genus, as already noticed in [2]. In particular it is still unknown whether $gr^*PB(\Sigma, n)$ is torsion-free or if $PB(\Sigma, n)$ is residually nilpotent, which would shed some light about the relevance of the filtration considered in this section.

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