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COMPTES RENDUS MATHEMATIQUE

Algebraic Geometry

Chow groups of surfaces with $h^{2,0} \leqslant 1$

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Abstract

We will investigate the geometry of rational equivalence classes of points on a surface S. We will show that if S is a general projective K3 surface then these equivalence classes are dense in the complex topology. We will also show that if S has the property that these equivalence classes are Zariski dense, then $h^{2,0}(S) \leqslant 1$. ***To cite this article: C. Maclean, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Les groupes de Chow des surfaces telles que $h^{2,0} \leqslant 1$. Nous considérons la géométrie des classes d'équivalence rationelle des points d'une surface S. Nous montrons que si S est une surface K3 générale, ces classes d'équivalence sont denses pour la topologie complexe. Nous montrons également que si S a la propriété que ces classes d'équivalence sont Zariski dense, alors $h^{2,0}(S) \leqslant 1$. ***Pour citer cet article : C. Maclean, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction and statement of results

The connection between the Chow group $\mathrm{CH}_0(S)$ of 0-cycles on a surface S and $h^{2,0}(S)$ has been an object of interest since Mumford's 1968 paper [6], in which he proved the following result.

Theorem 1.1 (Mumford). *If* $\mathrm{CH}_0(S)$ *is representable, then* $h^{2,0}(S) = 0$.

Bloch [1] conjectured that the converse is also true.

Conjecture 1 (Bloch). *If S is a smooth projective surface and $h^{2,0}(S) = 0$ then* $\mathrm{CH}_0(S)$ *is representable.*

Bloch, Kas and Liebermann proved the Bloch conjecture for surfaces not of general type in [2]. This conjecture has also been shown to hold for various surfaces of general type such that $h^{2,0}(S) = 0$ – see, for example, [9].

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Our aim is to show there is also a close connection between the condition $h^{2,0}(S) = 1$ and the geometry of 0-cycles on S. In particular, we will show the following result.

Theorem 1.2. *Let S be a general smooth projective* K3 *surface. Then for general $x \in S$, the set*

$$\{y \in S \mid y \equiv x\}$$

is dense in S (for the complex topology).

Here $\equiv$ denotes rational equivalence between points. We will also prove a partial converse to this result.

Theorem 1.3. *Let S be a smooth complex surface, such that for a generic point x of S the set*

$$\{y \in S \mid y \equiv x\}$$

is Zariski dense in S. Then $h^{2,0}(S) \leqslant 1$.

2. Proof of Theorem 1.2

The proof of this theorem relies on three fundamental facts:

1. If E is an elliptic curve and $x \in E$, then the set $\{y \in E \mid ny \equiv nx$ for some integer $n\}$ is dense in E.
2. There are many families of elliptic curves on a K3 surface.
3. By a theorem of Roitman's, [7], the Chow group of a K3 surface is torsion-free.

What we actually need to prove the theorem is two one-dimensional families of elliptic curves which intersect transversally. A sketch proof the existence of singular rational curves and families of singular elliptic curves can be found in the appendix to [5], attributed to Mumford and Bogolomov independently. Chen proved in [3] the following theorem.

Theorem 2.1 (Chen). *For any integers $n \geqslant 3$ and $d > 0$, the linear system $|\mathcal{O}_S(d)|$ on a general* K3 *surface S in $\mathbb{P}^n$ contains an irreducible nodal rational curve.*

The following proposition is an easy corollary of this (see the appendix to [5] or [4] (p. 70)).

Proposition 2.2. *The linear system $|\mathcal{O}_S(d)|$ on a general* K3 *surface S in $\mathbb{P}^n$ contains a* 1*-dimensional family of curves of geometric genus $\leqslant 1$ whose general element is irreducible and nodal.*

Indeed, the proposition follows from the theorem by a standard dimension count. If S is a K3 surface in $\mathbb{P}^N$ and the general element of $|\mathcal{O}_S(d)|$ is a smooth curve of genus g, then Saint-Donat calculated in [8] (p. 609) that the dimension of $|\mathcal{O}_S(d)|$ is g. However, the codimension of the space of nodal irreducible curves of geometric genus $\leqslant 1$ in M_g is $g - 1$. The proposition follows.

We now choose two distinct irreducible 1-dimensional families,

$$\pi_1 : F_1 \to B_1, \qquad \pi_2 : F_2 \to B_2$$

which are in the linear systems $|\mathcal{O}_S(1)|$ and $|\mathcal{O}_S(2)|$ respectively and whose general elements are integral nodal curves of geometric genus $\leqslant 1$. There are surjective maps $\phi_i : F_i \to S$.

We consider those $x \in S$ such that x is not contained in the image under ϕ_2 of any non-integral fibre of π_2. This is the only condition needed to prove the theorem for x.

Choose $y \in F_1$ such that $\phi_1(y) = x$ and denote $\pi_1(y)$ by z. Denote the curve $\pi_1^{-1}(z)$ by D. There is a surjective map from a nodal curve of genus $\leqslant 1$ to D, $r : \overline{D} \to D$.

Every component of $\overline{D}$ is of geometric genus $\leqslant 1$. There is a component of $\overline{D}$ which intersects $(\phi_1 \circ r)^{-1}(x)$ and whose image under $\phi_1 \circ r$ is not a single point. Denote the image of this component by E. Since E has a normalisation of genus $\leqslant 1$, the set

$$\bigl\{z \in E \text{ such that } z - x \text{ is torsion in } \mathrm{CH}^0(E)\bigr\}$$

is dense in E. By a result of Roitman's, [7], the torsion part of $\mathrm{CH}^0(S)$ is 0. Hence, the set

$$\bigl\{z \in E \text{ such that } z \equiv x \text{ in } \mathrm{CH}^0(S)\bigr\}$$

is dense in E.

Our strategy is as follows. The curve E is transverse to general elements of the family F_2. Consider the curves in the family F_2 which are elliptic or rational and meet E in a point rationally equivalent to x. The set of such curves is dense in F_2. If E_2 is such a curve then the set {points of E_2 rationally equivalent to x} is dense in E_2.

More precisely, consider the variety $V = \phi_2^{-1}(E)$ which parameterises points of intersection of E with a curve in the family F_2. The projection of V onto B_2 is surjective. Let S_E be the set

$$\bigl\{y \in E \text{ such that } y \equiv x \text{ in } \mathrm{CH}^0(S)\bigr\}.$$

The set S_E is dense in E for the complex topology. We denote by T the closure of

$$\bigl\{y \in S \text{ such that } y \equiv x \text{ in } \mathrm{CH}^0(S)\bigr\}.$$

We define $\widetilde{B}_2$ to be the open set in B_2 parameterising irreducible members of the family F_2. Consider

$$Z = \pi_2 \circ \phi_2^{-1}(S_E),$$

the set parameterising curves in the family F_2 which meet E in at least one point of $S(E)$. We denote by $\widetilde{Z}$ the set $Z \cap \widetilde{B}_2$. Once again, if $z \in \widetilde{Z}$, then the set

$$\bigl\{y \in F_{2,z} \text{ such that } y \equiv x \text{ in } \mathrm{CH}^0(S)\bigr\}$$

is dense in $F_{2,z}$, the fibre over z in F_2. Hence, T contains $\pi_2^{-1}(\widetilde{Z})$. We now need the following lemma.

Lemma 2.3. *The set Z is dense in B_2.*

Proof. There is a component C of V mapping surjectively to E. Since x is not contained in any non-integral fibre of π_2, and E is not an element of $|\mathcal{O}_S(2)|$ for degree reasons, $\pi_2 : C \to B_2$ is surjective. Since C is irreducible and $\phi_2|_C$ is surjective onto E $\phi_2|_C^{-1}(S(E))$ is dense in C. It follows that, since $\pi_2|_C$ is surjective and continuous, Z is dense in B_2. □

It immediately follows that T is dense in S. This completes the proof of Theorem 1.2.

3. Proof of Theorem 1.3

Now suppose that S satisfies the hypothesis that for general $x \in S$ the set $\{y \in S \mid x \equiv y \in \mathrm{CH}^0(S)\}$ is Zariski dense in S. We want to show that $h^{2,0}(S) \leqslant 1$. Mumford proved the following result in [6].

Theorem 3.1 (Mumford). *There exists a countable union of maps of reduced algebraic schemes $\phi_i : W_i \to S \times S$ such that the following hold.*

(1) *$x \equiv y$ if and only if there exists i such that $(x, y) \in \phi_i(W_i)$.*
(2) *Let pr^1 and pr^2 be the two projections from $S \times S$ onto S. Consider the maps*

$$\pi_i^1 \text{ and } \pi_i^2 : W_i \to S$$

given by $\pi_i^j = pr^j \circ \phi_i$. *We then have for any* 2*-form on* S, ω,

$$\pi_i^{1*}(\omega) = \pi_i^{2*}(\omega).$$

We may restrict ourselves to the case where the images of all the maps ϕ_i are of dimension $\leqslant 2$, since Mumford proved in [6] that

Proposition 3.2 (Mumford). *If there is an* i *such that the image of* ϕ_i *is of dimension* $\geqslant 3$ *then* $h^{2,0}(S) = 0$.

We now choose y such that

(1) $y \notin \pi_i^j(W_i)$ for any i such that $\dim(\operatorname{Im}\phi_i) \leqslant 1$.
(2) There do not exist x, i, j such that $(x, y) \in \operatorname{Im}(\phi_i)$ and π_i^j is not submersive at any point of $\phi_i^{-1}(x, y)$.
(3) The set $\{x \in S \mid y \equiv x\}$ is Zariski dense in S.

Since the varieties described in (1) and (2) are of dimension $\leqslant 1$ and, by assumption, (3) holds for general y, there exists such a y. The theorem follows from the following proposition.

Proposition 3.3. *There is no non-zero* 2*-form* ω *on* S *vanishing at* y.

Proof. Let ω be such a 2-form, and consider $x \in S$ such that $y \equiv x$. By the assumptions on y it follows that ω vanishes at x. Indeed, there is some W_i such that $(x, y) \in \phi_i(W_i)$. By assumption (2), there exists $p \in W_i$ such that $\phi_i(p) = (x, y)$ and π_i^1, π_i^2 are both submersive at p. We know that $\pi_i^{2*}(\omega)(p) = 0$ since $\omega(y) = 0$. It follows that $\pi_i^1 * (\omega)(p) = 0$. But by assumptions (1) and (2), this implies that $\omega(x) = 0$. Therefore, since the set of such points is Zariski dense, ω is identically 0. □

It follows immediately that $h^{2,0}(S) \leqslant 1$. This completes the proof of the theorem.

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