Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

C. R. Acad. Sci. Paris, Ser. I 337 (2003) 445–450

COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Monotonicity in integrodifferential equations

Jérôme Coville

Laboratoire Jacques Louis Lions, Université Paris VI, 175, rue du Chevaleret, 75013 Paris, France

Received and accepted 17 July 2003

Presented by Haïm Brezis

Abstract

We study the behavior of positive solutions of the Dirichlet problem $Lu = f(u)$ in Ω with $\Omega = (a, +\infty)$, where a can be $-\infty$, and L is an abstract operator which is non-increasing under translation and satisfies a strong maximum principle property. This covers the case of many integral operators. Under some assumptions on f (e.g., bistable, monostable), we show that any solution exhibits a monotone behavior. ***To cite this article: J. Coville, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Monotonie dans les équations intégro-différentielles. On présente plusieurs résultats concernant le comportement des solutions positives du problème de Dirichlet $Lu = f(u)$ sur un ouvert Ω, où $\Omega = (a, +\infty)$ avec a pouvant être égale à $-\infty$. Ici, L est un opérateur vérifiant un principe du maximum fort ainsi qu'une propriété de décroissance par translation. Nos résultats couvrent le cas d'opérateurs intégraux. On établit le caractère monotone des solutions pour certaines classes de nonlinéarités f. ***Pour citer cet article : J. Coville, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

On étudie le comportement des solutions positives des deux classes de problèmes suivants :

$$Lu = -f(u) \quad \text{sur } \mathbb{R}, \tag{1}$$

$$u(x) \to 0 \quad \text{quand } x \to -\infty, \tag{2}$$

$$u(x) \to 1 \quad \text{quand } x \to +\infty \tag{3}$$

et

$$Lu = -f(u) \quad \text{sur } (a, +\infty), \tag{4}$$

$$0 \leqslant u(a) < u(x) < 1, \tag{5}$$

$$u(x) \to 1 \quad \text{quand } x \to +\infty. \tag{6}$$

E-mail address: coville@ann.jussieu.fr (J. Coville).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2003.07.005

On suppose que L est un opérateur qui transforme les fonctions regulières définies sur $\mathbb{R}$ en des fonctions continues de $\mathbb{R}$. Plus précisément L vérifie :

$$\exists k \in \mathbb{N} \quad L : C^k(\mathbb{R}) \to C^0(\mathbb{R}).$$

On suppose aussi que L vérifie les propriétés suivantes

– Soit $U_h(\cdot) := U(\cdot + h)$, alors pour tout $h > 0$ on a $L[U_h](x) \leqslant L[U](x + h)$ pour tout $x \in \mathbb{R}$.
– Si v est une constante positive alors $L[v](x) \leqslant 0$ pour tout $x \in \mathbb{R}$.

On suppose par ailleurs que L vérifie la propriété suivante qui est une version renforcée du principe du maximum fort :

Hypothèse 1 (Principe du Maximum). *Soit u une fonction régulière, si u atteint un minimum* (*resp. maximum*) *en un point x de $\mathbb{R}$ alors on a l'alternative suivante* :

– *Soit* $L[u](x) > 0$ (*resp.* <0) ;
– *Soit* $L[u](x) \leqslant 0$ (*resp.* $\geqslant 0$) *et u est identiquement égale à une constante.*

Remarque 1. Dans l'énoncé du principe du maximum on peut remplacer $\mathbb{R}$ par un ouvert $\Omega = (a, +\infty)$.

Remarque 2. Le Laplacien ne vérifie pas cette version renforcée du principe du maximum fort. Par contre, les opérateurs du type $Lu := \int_{\mathbb{R}} J(x - y)(u(y) - u(x))\,\mathrm{d}y$ le vérifient en supposant que J est une fonction positive, paire, continue et intégrable.

Par ailleurs, on suppose que les nonlinéarités f sont des fonctions régulières et choisies de telle sorte que les solutions u vérifient $0 < u < 1$. On établit trois théorèmes correspondant à trois types de nonlinéarités f :

Théorème 0.1. *Soit $f \in C^1((0,1))$ telle que pour un $\rho > 0$ on ait $f|_{(0,\rho)} \equiv 0$, $f|_{(\rho,1)} > 0$, $f(0) = f(1) = 0$ et $f'(1) < 0$. Alors toute solution positive et régulière de* (1)–(3) *est strictement croissante.*

Théorème 0.2. *Soit $f \in C^1((0,1))$ telle que $f|_{(0,1)} > 0$, $f(0) = f(1) = 0$ et $f'(1) < 0$. Si u est une solution positive, régulière de* (1)–(3) *et strictement croissante sur $(-\infty, -M)$ pour un M positif alors u est strictement croissante sur $\mathbb{R}$.*

Théorème 0.3. *Soit $f \in C^1((0,1))$ telle que $f(0) = f(1) = 0$ et $f'(1) < 0$. Alors toute solution positive et régulière de* (4)–(6) *est strictement croissante.*

Remarque 3. Pour les nonlinearités f considérées, il suffit d'étendre par 0 les fonctions f en dehors de l'intervalle (0,1) pour satisfaire la condition $0 < u < 1$.

Les techniques employées, peuvent s'appliquer en dimension supérieur, notament on obtient très facilement les théorèmes équivalents aux Théorèmes 0.1 et 0.2 pour des problèmes posés sur des cylindres infinis $\Sigma = \omega \times \mathbb{R}$ du type :

$$Lu = -f(u) \quad \text{sur } \Sigma, \tag{7}$$

$$u(x', x_N) \to 0 \quad \text{quand } x_N \to -\infty, \tag{8}$$

$$u(x', x_N) \to 1 \quad \text{quand } x_N \to +\infty, \tag{9}$$

où ω est un ouvert borné de $\mathbb{R}^{N-1}$ et les convergences sont supposées uniformes par rapport à x'.

Ces résultats étendent aux cas d'opérateurs non locaux et dans un cadre générale des résultats de monotonie de fronts progressifs [1–3,6–8].

1. Main results

We study the behavior of solutions to the following Dirichlet problems:

$$Lu = -f(u) \quad \text{on } \mathbb{R}, \tag{10}$$

$$u(x) \to 0 \quad \text{as } x \to -\infty, \tag{11}$$

$$u(x) \to 1 \quad \text{as } x \to +\infty \tag{12}$$

and

$$Lu = -f(u) \quad \text{on } (a, +\infty), \tag{13}$$

$$0 \leqslant u(a) < u(x) < 1, \tag{14}$$

$$u(x) \to 1 \quad \text{as } x \to +\infty, \tag{15}$$

where L is an operator which maps smooth functions defined on $\mathbb{R}$ to continuous function. More precisely, L satisfies:

$$\exists k \in \mathbb{N} \quad L : C^k(\mathbb{R}) \to C^0(\mathbb{R}).$$

We assume further that L satisfies the following property:

- Let $U_h(\cdot) := U(\cdot + h)$, then for all $h > 0$ we have $L[U_h](x) = L[U](x + h)\ \ \forall x \in \mathbb{R}$.
- Let v a positive constant then we have $L[v](x) \leqslant 0$.

The main assumption on L is the following maximum principle:

Assumption 1 (*Maximum Principle*). Let u be a smooth function, if u achieves a minimum (resp. maximum) at some point x in $\mathbb{R}$ then the following holds:

- Either $L[u](x) > 0$ (resp. < 0);
- Or $L[u](x) \leqslant 0$ (resp. $\geqslant 0$) and u is identically constant.

Remark 1. Note that we can state a maximum principle for any open set $\Omega = (a, +\infty)$ as well.

Remark 2. $L = \Delta$ does not satisfy this strengthened maximum principle. On the other hand, it is verified by the operator $Lu := \int_{\mathbb{R}} J(x - y)(u(y) - u(x))\,\mathrm{d}y$ where J is a positive even continuous integrable function.

In addition, we assume that the nonlinearities f are smooth and such that the solutions u satisfy $0 < u < 1$. We prove three theorems which correspond to three different kinds of nonlinearity f:

Theorem 1.1. *Let $f \in C^1((0,1))$ such that for some $\rho > 0$ we have $f|_{(0,\rho)} \equiv 0,\ f|_{(\rho,1)} > 0,\ f(0) = f(1) = 0$ and $f'(1) < 0$. Then any positive, smooth solution of* (10)–(12) *is monotone increasing.*

Theorem 1.2. *Let $f \in C^1((0,1))$ such that $f|_{(0,1)} > 0$, $f(0) = f(1) = 0$ and $f'(1) < 0$. Then any positive smooth solution of* (10)–(12)*, which is monotone increasing near $-\infty$, is monotone increasing on all $\mathbb{R}$.*

Theorem 1.3. *Let $f \in C^1((0,1))$ such that $f(0) = f(1) = 0$ and $f'(1) < 0$. Then any nonnegative, smooth solution u of* (13)–(15) *is monotone increasing.*

Remark 3. The condition $0 < u < 1$ is easily obtained in our three cases. We just have to extend the nonlinearity f by 0 outside the interval (0,1) and use the maximum principle.

The tools that we developed can be applied in other situations, namely we get theorems like 1.1 and 1.2 for problems of the following type:

$$Lu = -f(u) \quad \text{on } \Sigma, \tag{16}$$

$$u(x', x_N) \to 0 \quad \text{when } x_N \to -\infty, \tag{17}$$

$$u(x', x_N) \to 1 \quad \text{when } x_N \to +\infty, \tag{18}$$

where $\Sigma = \omega \times \mathbb{R}$ is an infinite cylinder and ω a bounded open subset of $\mathbb{R}^{N-1}$. The limits are taken uniformly with respect to x'. Theses results extended previous results on monotonicity of travelling fronts [1–3,6–8] to the case of nonlocal integral operator. We now give a sketch of our proof.

2. Sketch of the proofs

The proofs of these three theorems follow the same scheme. So we will only describe the proof of Theorem 1.1. We take ξ, instead of x, as space variable. We use the sliding idea which was introduced by Berestycki and Nirenberg [2] and developed by Berestycki and Hamel [1]. We will compare, via the maximum principle, the function u and its translated $u_\tau := u(\cdot + \tau)$.

We first show a lemma which says that we can order, at least for some positive τ, u and u_τ.

Lemma 2.1. *Let (u) be a positive smooth solution of* (10)–(12)*. Then there exists a positive τ such that $u \leqslant u_\tau$.*

Then we prove a proposition which asserts that if we can order u_τ and u, for one τ then we can order $u_{\tilde\tau}$ and u with $\tilde\tau \geqslant \tau$. Namely, we have

Proposition 2.1. *Let (u) be a positive solution of* (10)–(12)*. If there exists $\tau > 0$ such that $u \leqslant u_\tau$ then $u \leqslant u_{\tilde\tau}$ $\forall \tilde\tau \geqslant \tau$*

The proof of Proposition 2.1 is based on the following two technical lemmas.

Lemma 2.2. *Let (u) be a positive solution of* (10)–(12) *and $\tau > 0$ such that $u \leqslant u_\tau$. Then, we have $u(\xi) < u(\xi + \tau)$ $\forall \xi \in \mathbb{R}$.*

Lemma 2.3. *Let (u) be a positive solution of* (10)–(12) *and $\tau > 0$ such that $u(\xi) < u(\xi + \tau)$ $\forall \xi \in \mathbb{R}$. Then there exists $\varepsilon_0(\tau) > 0$ such that for all $\tilde\tau \in [\tau, \tau + \varepsilon_0]$, we have*

$$u(\xi) < u(\xi + \tilde\tau) \quad \forall \xi \in \mathbb{R}. \tag{19}$$

Now from Proposition 2.1, we can seek for the minimal τ which preserve the order of u and u_τ and show that this infimum can not be positive. More precisely, we define

$$\tau^* = \inf\{\tau > 0 \mid \forall \tilde{\tau} \geqslant \tau,\ u(\xi + \tilde{\tau}) \geqslant u(\xi)\ \forall \xi \in \mathbb{R}\}. \tag{20}$$

We want to show that $\tau^* \leqslant 0$. This is done by a contradiction argument. If not then we can construct a $\tau < \tau^*$ which satisfies $u \leqslant u_\tau$ and by Proposition 2.1 contradict the definition of τ^* (see [4] for details). Therefore, the function u is nondecreasing. Applying Lemma 2.2 we conclude that u is indeed increasing.

Proof of Proposition 2.1. We know from Lemma 2.1 that we can find a positive τ, such that,

$$u(\xi) \leqslant u(\xi + \tau) \quad \forall \xi \in \mathbb{R}.$$

Therefore from Lemmas 2.2 and 2.3, we can construct an interval $[\tau, \tau + \varepsilon]$, such that for all $\tilde{\tau} \in [\tau, \tau + \varepsilon]$ we have

$$u(\xi) \leqslant u(\xi + \tilde{\tau}) \quad \forall \xi \in \mathbb{R}.$$

Let us define the following quantity,

$$\bar{\tau} = \sup\{\tilde{\tau} \mid \forall \hat{\tau} \in [\tau, \tilde{\tau}],\ u(\xi) \leqslant u(\xi + \hat{\tau})\ \forall \xi \in \mathbb{R}\}. \tag{21}$$

We claim that $\bar{\tau} = +\infty$. If not, $\bar{\tau} < +\infty$ and by continuity we have

$$u(\xi) \leqslant u(\xi + \bar{\tau}) \quad \forall \xi \in \mathbb{R}. \tag{22}$$

Recall that from the definition of $\bar{\tau}$, we have

$$\forall \hat{\tau} \in [\tau, \bar{\tau}] \quad u(\xi) \leqslant u(\xi + \hat{\tau}) \quad \forall \xi \in \mathbb{R}. \tag{23}$$

Therefore to get a contradiction, it is sufficient to construct ε_0 such that for all $\varepsilon \in [0, \varepsilon_0]$, we have

$$u(\xi) \leqslant u\big(\xi + (\bar{\tau} + \varepsilon)\big) \quad \forall \xi \in \mathbb{R}. \tag{24}$$

Since $\bar{\tau} > 0$, we can apply Lemma 2.2 to have,

$$u(\xi) < u(\xi + \bar{\tau}) \quad \forall \xi \in \mathbb{R}. \tag{25}$$

We can now apply Lemma 2.3, to find the desired $\varepsilon > 0$. Therefore, from the definition of $\bar{\tau}$ we get

$$\forall \hat{\tau} \in [\tau, +\infty], \quad u(\xi) \leqslant u(\xi + \hat{\tau}) \quad \forall \xi \in \mathbb{R}.$$

This complete the proof of Proposition 2.1. □

The details of these arguments will appear in [4,5].

Acknowledgements

I would like to thank Henri Berestycki for his precious advice and Pascal Autissier, Louis Dupaigne, Augusto Ponce for entlightning discussions on this subject.

References

[1] H. Berestycki, F. Hamel, Front propagation in periodic excitable media, Comm. Pure Appl. Math. 55 (8) (2002) 949–1032.
[2] H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, Bol. Soc. Brasil. Math. 22 (1) (1991) 1–37.
[3] H. Berestycki, B. Nikolaenko, B. Sheurer, Travelling wave solutions to combustion models and their singular limits, SIAM J. Math. Anal. 16 (6) (1985) 1207–1242.

[4] J. Coville, On monotone behavior of solution of nonlocal reaction–diffusion equation, Publication du laboratoire Jacques-Louis Lions R03006.
[5] J. Coville, L. Dupaigne, Min–Max formula for the minimal speed of an integrodifferential reaction–diffusion equation, Preprint.
[6] A. Farina, Monotonicity one-dimensional symmetry for the solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$ with possibly discontinuous nonlinearity, Adv. Math. Sci. Appl. 11 (2) (2001) 811–834.
[7] P.C. Fife, Mathematical Aspects of Reacting and Diffusing Systems, in: Lecture Notes in Biomath., Vol. 28, Springer-Verlag, Berlin, 1979.
[8] P. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equation to travelling front solutions, Arch. Rational Mech. Anal. 65 (1977) 335–361.