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Mathematical Analysis

Complex interpolation between two weighted Bergman spaces on tubes over symmetric cones

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Abstract

We prove that the complex interpolation space $[A_\nu^{p_0}, A_\nu^{p_1}]_\theta$, $0<\theta<1$, between two weighted Bergman spaces $A_\nu^{p_0}$ and $A_\nu^{p_1}$ on the tube in $\mathbb{C}^n$, $n \geqslant 3$, over an irreducible symmetric cone of $\mathbb{R}^n$ is the weighted Bergman space A_ν^p with $1/p = (1-\theta)/p_0 + \theta/p_1$. Here, $\nu > n/r - 1$ and $1 \leqslant p_0 < p_1 < 2 + \nu/(n/r-1)$ where r denotes the rank of the cone. We then construct an analytic family of operators and an atomic decomposition of functions, which are related to this interpolation result. ***To cite this article: D. Békollé et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Interpolation complexe entre deux espaces de Bergman à poids dans des tubes au-dessus de cônes symétriques. Nous donnons une démonstration du fait que par la méthode complexe, l'espace d'interpolation $[A_\nu^{p_0}, A_\nu^{p_1}]_\theta$, $0<\theta<1$, entre deux espaces de Bergman à poids $A_\nu^{p_0}$ et $A_\nu^{p_1}$ est l'espace de Bergman à poids A_ν^p, avec $1/p = (1-\theta)/p_0 + \theta/p_1$, dans le tube de $\mathbb{C}^n$, $n \geqslant 3$, au-dessus d'un cône symétrique irréductible de $\mathbb{R}^n$. Ici, $\nu > n/r - 1$, $1 \leqslant p_0 < p_1 < 2 + \nu/(n/r-1)$, où r désigne le rang du cône. Nous construisons ensuite une famille analytique d'opérateurs et une décomposition atomique de fonctions, qui sont en relation avec ce résultat d'interpolation. ***Pour citer cet article : D. Békollé et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soit T_Ω le tube de $\mathbb{C}^n$, $n \geqslant 3$, au-dessus d'un cône symétrique irréductible Ω de $\mathbb{R}^n$. Comme dans [4], nous désignons par r le rang du cône Ω, et par $\Delta(x)$ le déterminant de $x \in \mathbb{R}^n$. Pour $1 \leqslant p < \infty$ and $\nu > n/r - 1$, on écrit $L_\nu^p = L^p(T_\Omega, \Delta(y)^{\nu - n/r}\,\mathrm{d}x\,\mathrm{d}y)$. L'espace de Bergman à poids A_ν^p est le sous-espace fermé de l'espace de Banach L_ν^p formé par les fonctions holomorphes.

Nous donnons d'abord une démonstration du théorème suivant :

Théorème. *On suppose que* $1 \leqslant p_0 < p_1 < 2 + \frac{\nu}{n/r-1}$, $0<\theta<1$. *Alors l'espace d'interpolation* $[A_\nu^{p_0}, A_\nu^{p_1}]_\theta$ *est égal à* A_ν^p, *avec* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *et les normes sur les deux espaces sont équivalentes.*

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Dans la suite, nous posons $Q_\nu = 1 + \frac{\nu}{n/r-1}$. Il suffit de démontrer ce théorème pour $p_0 = 1$ et $2 < p_1 < Q_\nu + 1$. Quel que soit $\alpha > \frac{n}{r} - 1$, nous désignons par P_α le projecteur de Bergman relatif à la mesure $\Delta(y)^{\alpha-n/r}\,\mathrm{d}x\,\mathrm{d}y$, i.e., le projecteur orthogonal de l'espace de Hilbert L^2_α sur son sous-espace fermé A^2_α. On déduit d'un résultat de [3] que l'opérateur $P_{2\nu+n/r}$ se prolonge en un projecteur continu de L^p_ν sur A^p_ν si $1 \leqslant p < Q_\nu$, et il s'ensuit que si $1 < q_3 < Q_\nu$, $[A^1_\nu, A^{q_3}_\nu]_\theta = A^{q_2}_\nu$, où $\frac{1}{q_2} = 1 - \theta + \frac{\theta}{q_3}$. D'autre part, il est démontré dans [2] que P_ν se prolonge en un projecteur continu de L^p_ν sur A^p_ν si $(Q_\nu + 1)' < p < Q_\nu + 1$. Il s'ensuit que si $(Q_\nu + 1)' < q_2 < q_4 < Q_\nu + 1$, $[A^{q_2}_\nu, A^{q_4}_\nu]_\theta = A^{q_3}_\nu$, où $\frac{1}{q_3} = \frac{1-\theta}{q_2} + \frac{\theta}{q_4}$. On conclut alors en utilisant le théorème de réitération de Wolff (cf. [6]).

En même temps, pour établir l'inclusion $A^p_\nu \subset [A^1_\nu, A^{p_1}_\nu]_\theta$, il est naturel de chercher une application holomorphe explicite dans la bande $S = \{z \in \mathbb{C}:\ 0 \leqslant \Re\mathrm{e}\, z \leqslant 1\}$, à valeurs dans $A^{p_0}_\nu + A^1_\nu$, qui coïncide en θ avec une fonction f donnée dans A^p_ν. Pour cela, nous nous basons sur une décomposition atomique adaptée de l'espace de Bergman. Plus précisément, soit $\{w_j = u_j + \mathrm{i}v_j\}_{j\in\mathbb{N}^*}$ un δ-réseau du tube, $0 < \delta < 1$. On dit qu'une suite $\{\lambda_j\}_{j\in\mathbb{N}^*}$ appartient à l^p_ν si la somme $\sum_j |\lambda_j|^p \Delta(v_j)^{\nu+n/r}$ est finie. Nous considérons la famille analytique $\{T_z\}_{z\in S}$ d'opérateurs définis sur l'espace des suites finies de nombres complexes, à valeurs dans l'espace des fonctions mesurables dans le tube, comme suit : $T_z(\{\lambda_j\}) = c_{\nu+(\nu+n/r)z}\,\mathrm{e}^{(z-\theta)^2} \sum_j \lambda_j \Delta(v_j)^{\nu+(\nu+n/r)z+n/r} \Delta^{-\nu-(\nu+n/r)z-n/r}((\cdot - \overline{w}_j)/\mathrm{i})$. Nous démontrons le théorème suivant :

Théorème. *Soient $\theta \in]0,1[$, $1 \leqslant p_1 < Q_\nu + 1$. On pose $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$ et $\alpha(z) = p(1 - z + \frac{z}{p_1})$. Alors quel que soit $\{\lambda_j\} \in l^p_\nu$, si l'on définit*

$$\lambda_j(z) = \begin{cases} |\lambda_j|^{\alpha(z)} \frac{\lambda_j}{|\lambda_j|} & \text{si } \lambda_j \neq 0, \\ 0 & \text{sinon} \end{cases}$$

l'application $f(z) = T_z(\{\lambda_j(z)\})$ est une application holomorphe de S dans $A^1_\nu + A^{p_1}_\nu$. De plus, quel que soit $g \in A^p_\nu$, il existe une suite $\{\lambda_j\} \in l^p_\nu$ telle que g s'obtient exactement sous la forme $g = f(\theta)$.

1. Introduction

Let n be an integer such that $n \geqslant 3$. We denote by Ω an irreducible symmetric cone in $\mathbb{R}^n$. Referring to [4], we denote by $(\cdot|\cdot)$ the canonical scalar product in $\mathbb{R}^n$, by r the rank of the cone Ω, by $\Delta(x)$ the determinant of $x \in \mathbb{R}^n$ and by $\mathbf{e}$ the identity element of $\mathbb{R}^n$ regarded as a Euclidean Jordan algebra. The Gamma function of Ω is defined by $\Gamma_\Omega(\lambda) = \int_\Omega \mathrm{e}^{-(x|\mathbf{e})} \Delta(x)^{\lambda-n/r}\,\mathrm{d}x$ with $\lambda \in \mathbb{C}$ satisfying $\Re\mathrm{e}\,\lambda > \frac{n}{r} - 1$. It is well known that for $y \in \Omega$ and $\Re\mathrm{e}\,\lambda > \frac{n}{r} - 1$,

$$\int_\Omega \mathrm{e}^{-(x|y)} \Delta(x)^{\lambda-n/r}\,\mathrm{d}x = \Gamma_\Omega(\lambda)\Delta(y)^{-\lambda}. \tag{1}$$

Explicitly, $\Gamma_\Omega(\lambda) = \pi^{n/r-1}\Gamma(\lambda)\Gamma(\lambda - \frac{d}{2})\cdots\Gamma(\lambda - (r-1)\frac{d}{2})$, where $d = 2\frac{n/r-1}{r-1}$. We shall denote $T_\Omega = \mathbb{R}^n + \mathrm{i}\Omega$ the tube domain over the cone Ω. If we fix $\lambda \in \mathbb{C}$ such that $\Re\mathrm{e}\,\lambda > \frac{n}{r} - 1$, then the integral function defined on T_Ω by $\zeta \mapsto \frac{1}{\Gamma_\Omega(\lambda)} \int_\Omega \mathrm{e}^{-(x|\zeta/\mathrm{i})} \Delta(x)^{\lambda-n/r}\,\mathrm{d}x$ is absolutely convergent and defines a holomorphic function on T_Ω. By (1), this holomorphic function is an extension of the function $\Delta(y)^{-\lambda}$ defined on Ω, so we shall denote it by $\Delta^{-\lambda}(\frac{\zeta}{\mathrm{i}})$. For $1 \leqslant p < \infty$ and $\nu > \frac{n}{r} - 1$, we write $L^p_\nu = L^p(T_\Omega, \Delta(y)^{\nu-n/r}\,\mathrm{d}x\,\mathrm{d}y)$. The weighted Bergman space A^p_ν is the closed subspace of the Banach space L^p_ν consisting of holomorphic functions. The weighted Bergman projector P_ν of T_Ω is the orthogonal projector of the Hilbert space L^2_ν onto its closed subspace A^2_ν. It is well known that for every $f \in L^2_\nu$, $P_\nu f(\zeta) = \int_{T_\Omega} B_\nu(\zeta, u + \mathrm{i}v) f(u + \mathrm{i}v)\Delta(v)^{\nu-n/r}\,\mathrm{d}u\,\mathrm{d}v$, where $B_\nu(\zeta, w) = c_\nu \Delta^{-\nu-n/r}(\frac{\zeta-\overline{w}}{\mathrm{i}})$ is the weighted Bergman kernel of T_Ω, with $c_\nu = (2\pi)^{-n} 2^{r\nu} [\Gamma_\Omega(\nu)]^{-1} \Gamma_\Omega(\nu + \frac{n}{r})$ (cf. [3]). In the sequel, for every $\gamma \in \mathbb{C}$ such that $\Re\mathrm{e}\,\gamma > \frac{n}{r} - 1$, we adopt the notation $c_\gamma = 2^{r\gamma}(2\pi)^{-n}[\Gamma_\Omega(\gamma)]^{-1}\Gamma_\Omega(\gamma + \frac{n}{r})$.

Moreover, it can be shown that for every $\gamma \in \mathbb{C}$ such that $\Re\mathrm{e}\,\gamma > \frac{n}{r} - 1$, the operator P_γ defined on $L^2_{\Re\mathrm{e}\,\gamma}$ by $P_\gamma f(\zeta) = c_\gamma \int_{T_\Omega} \Delta^{-\gamma-n/r}((\zeta - u + \mathrm{i}v)/\mathrm{i}) f(u+\mathrm{i}v)\Delta(v)^{\gamma-n/r}\,\mathrm{d}u\,\mathrm{d}v$, is a bounded operator from $L^2_{\Re\mathrm{e}\,\gamma}$ to $A^2_{\Re\mathrm{e}\,\gamma}$.

In the sequel, we write $Q_\nu = 1 + \frac{\nu}{n/r-1}$. We start with the following theorem:

Theorem 1.1. *Suppose* $1 \leqslant p_0 < p_1 < Q_\nu + 1$ *and* $0 < \theta < 1$. *Then* $[A_\nu^{p_0}, A_\nu^{p_1}]_\theta = A_\nu^p$ *with* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *and the norms on the two spaces are equivalent.*

For a proof, it suffices to take $p_0 = 1$ and $2 < p_1 < Q_\nu + 1$. Let $A_1 = A_\nu^1$ and $A_j = A_\nu^{q_j}$, $j = 2, 3, 4$, with $1 \leqslant q_j < \infty$. We point out that $A_1 \cap A_4$ is a dense subspace of the spaces A_2 and A_3 (cf. [2] and [3]). Furthermore, we can deduce from a result of [3] that the operator $P_{2\nu+n/r}$ extends to a bounded projector from L_ν^p to A_ν^p when $1 \leqslant p < Q_\nu$. As a consequence, if $1 < q_3 < Q_\nu$, the complex interpolation space $[A_1, A_3]_\theta$ is equal to A_2 with $\frac{1}{q_2} = 1 - \theta + \frac{\theta}{q_3}$. On the other hand, it was proved in [2] that P_ν extends to a bounded projector from L_ν^p to A_ν^p when $(Q_\nu + 1)' < p < Q_\nu + 1$. Hence, if $(Q_\nu + 1)' < q_2 < q_4 < Q_\nu + 1$, then $[A_2, A_4]_\theta = A_3$ with $\frac{1}{q_3} = \frac{1-\theta}{q_2} + \frac{\theta}{q_4}$. The conclusion then follows using Wolff's abstract reiteration theorem (cf. [6]).

However, to get the continuous inclusion $A_\nu^p \subset [A_\nu^{p_0}, A_\nu^{p_1}]_\theta$, with $p_0, p_1 \in [1, Q_\nu + 1)$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $0 < \theta < 1$, it is natural to look for an explicit holomorphic mapping on the strip $S = \{z \in \mathbb{C}\colon\ 0 \leqslant \Re\mathrm{e}\,z \leqslant 1\}$, with values in $A_\nu^{p_0} + A_\nu^{p_1}$, which coincides at θ with a given function $f \in A_\nu^p$ in such a way that $f \in [A_\nu^{p_0}, A_\nu^{p_1}]_\theta$. Our result (see Theorem 4.1 below for a precise statement) will rely on an adapted atomic decomposition of functions in the Bergman space A_ν^p.

2. L^p-boundedness and surjectivity of an analytic family of operators

In the sequel, we keep $\theta \in (0, 1)$ fixed. Let $S = \{z \in \mathbb{C}\colon\ 0 \leqslant \Re\mathrm{e}\,z \leqslant 1\}$. We shall consider the analytic family of linear operators $\{\mathsf{P}_z\}_{z\in S}$, mapping the space of simple functions in L_ν^1 into the space of measurable functions on T_Ω, defined as follows:

$$\mathsf{P}_z f(\zeta) = \mathrm{e}^{(z-\theta)^2} c_{\nu+(\nu+n/r)z} \int\limits_{T_\Omega} \Delta^{-\nu-(\nu+n/r)z-n/r}\left(\frac{\zeta - u + \mathrm{i}v}{\mathrm{i}}\right) f(u+\mathrm{i}v)\Delta(v)^{\nu+(\nu+n/r)z-n/r}\,\mathrm{d}u\,\mathrm{d}v.$$

The analytic family $\{\mathsf{P}_z\}_{z\in S}$ is admissible in the sense of [5], p. 205.

Theorem 2.1. *Let* $t \in \mathbb{R}$, $2 < q_0 < Q_\nu + 1$ *and* $q_1 = 1$.

(i) *For* $k = 0, 1$, *the operator* $\mathsf{P}_{k+\mathrm{i}t}$ *is bounded from* $L_\nu^{q_k}$ *to* $A_\nu^{q_k}$ *and* $\|\mathsf{P}_{k+\mathrm{i}t} f\|_{A_\nu^{q_k}} \leqslant m_k \|f\|_{L_\nu^{q_k}}$ *where* m_k *is a constant independent of* t.

(ii) *For* $\frac{1}{q} = 1 - \theta + \frac{\theta}{q_0}$, *the operator* P_θ *is a bounded projector from* L_ν^q *to* A_ν^q.

Proof. (i) For $k = 1$, one has $|\mathsf{P}_{1+\mathrm{i}t} f(\zeta)| \leqslant \mathrm{e}^{(1-\theta)^2 - t^2 + \pi(\nu+n/r)|t|} |c_{\nu+(\nu+n/r)(1+\mathrm{i}t)}| \Lambda(|f|)(\zeta)$, where $\Lambda g(\zeta) = \int_{T_\Omega} |\Delta^{-2\nu-2n/r}((\zeta - u + \mathrm{i}v)/\mathrm{i})| g(u+\mathrm{i}v)\Delta(v)^{2\nu}\,\mathrm{d}u\,\mathrm{d}v$. The positive integral operator Λ is bounded on L_ν^1 because for every $u + \mathrm{i}v \in T_\Omega$, $\int_{T_\Omega} |\Delta^{-2\nu-2n/r}((\sigma - u + \mathrm{i}(\tau + v))/\mathrm{i})|\Delta(\tau)^{\nu-n/r}\,\mathrm{d}\sigma\,\mathrm{d}\tau \leqslant c\Delta^{-\nu-n/r}(v)$. This is proved in [3]. Moreover, it is easy to obtain that $\|\mathsf{P}_{1+\mathrm{i}t}\|$ is bounded by a constant independent of t.

In the case where $k = 0$, we write $L_\nu^{p,q} = L^q(L^p(\mathsf{R}^n, \mathrm{d}u), \Delta^{\nu-n/r}(v)\,\mathrm{d}v)$, $1 \leqslant p, q \leqslant \infty$, and we denote by $A_\nu^{p,q}$ the mixed norm weighted Bergman space which is the closed subspace of $L_\nu^{p,q}$ consisting of holomorphic functions. The operator $\mathsf{P}_{\mathrm{i}t}$ is bounded from L_ν^{∞,r_1} to A_ν^{∞,r_1} with its operator norm bounded by a constant independent of t, if $Q_\nu' < r_1 < Q_\nu$ (see [3]). On the other hand, one shows that $\|\mathsf{P}_{\mathrm{i}t}\|_{A_\nu^{2,r_0}} \leqslant c_\nu^{-1} |c_{\nu+\mathrm{i}(\nu+n/r)t}| \|R_{\mathrm{i}(\nu+n/r)t} g\|_{A_\nu^{2,r_0}}$, where $g(\zeta) = f(\zeta)\Delta^{\mathrm{i}(\nu+n/r)t}(\Im\mathrm{m}\,\zeta)$, and for all $g \in L_\nu^2$ and $\Re\mathrm{e}\,\alpha > -\frac{\nu+1}{2}$,

$R_\alpha g(\zeta) = c_\nu \int_{T_\Omega} \Delta^{-\nu-n/r-\alpha}((\zeta - u + \mathrm{i}v)/\mathrm{i}) g(u+\mathrm{i}v) \Delta^{\nu-n/r}(v)\,\mathrm{d}u\,\mathrm{d}v$. If we denote by Θ_α the restriction of R_α to A^2_ν, then $R_{\mathrm{i}(\nu+n/r)t} = e_{\nu,\gamma,t}\Theta_{\mathrm{i}(\nu+n/r)t-\gamma} \circ R_\gamma$, where $\frac{n/r-1}{2} < \gamma < \frac{\nu+1}{2}$ and

$$e_{\nu,\gamma,t} = \frac{\Gamma_\Omega(\nu+n/r)\Gamma_\Omega(\nu+n/r+\mathrm{i}(\nu+n/r)t)}{\Gamma_\Omega(\nu+\gamma+n/r)\Gamma_\Omega(\nu-\gamma+n/r+\mathrm{i}(\nu+n/r)t)}.$$

Using results from [2] and [3], it can be shown that the operators R_γ and $\Theta_{\mathrm{i}(\nu+n/r)t-\gamma}$ are respectively bounded from L^{2,r_0}_ν to $A^{2,r_0}_{\nu+\gamma r_0}$ and from $A^{2,r_0}_{\nu+\gamma r_0}$ to A^{2,r_0}_ν if $Q'_\nu < r_0 < 2Q_\nu$. Moreover, $\|\Theta_{\mathrm{i}(\nu+n/r)t-\gamma}\| \leqslant c|\beta_{\nu,\gamma,t}|$, where $\beta_{\nu,\gamma,t} = \frac{\Gamma_\Omega(\nu+n/r)}{\Gamma_\Omega(\nu-\gamma+n/r+\mathrm{i}(\nu+n/r)t)}$. These results were proved in [1] for the particular case of the tube over the Lorentz cone. It then follows that for those values of r_0, $\|\mathsf{P}_{\mathrm{i}t} f\|_{A^{2,r_0}_\nu} \leqslant K\|f\|_{L^{2,r_0}_\nu}$, where the constant K does not depend on t. The announced result follows from the fact that $[L^{\infty,r_1}_\nu, L^{2,r_0}_\nu]_\varphi = L^{q_0}_\nu$ for some $\varphi \in (0,1)$.

(ii) The boundedness result for P_θ follows from (i) through the interpolation of the analytic family $\{\mathsf{P}_z\}_{z\in S}$ of operators (see [5], pp. 205–207). □

3. l^p-boundedness of an analytic family of atomic decomposition operators

Definition 3.1. A sequence $\{w_j\}_{j\in\mathbb{N}}$ is called a δ-lattice in T_Ω, $0<\delta<1$, if: (1) the Bergman balls with center w_j and radius $\frac{\delta}{2}$ are pairwise disjoint; (2) the Bergman balls with center w_j and radius δ form a cover of T_Ω with finite overlapping, i.e., there is a positive integer N such that each point of T_Ω belongs to at most N of these balls.

The existence of a δ-lattice in T_Ω is proved in [3], p. 66.

In the sequel, we fix a δ-lattice $\{w_j = u_j + \mathrm{i}v_j\}_{j\in\mathbb{N}}$. We say that a sequence $\{\lambda_j\}$ of complex numbers belongs to l^p_ν if the sum $\sum_j |\lambda_j|^p \Delta(v_j)^{\nu+n/r}$ is finite. On the other hand, the topological dual space of a normed space A will be denoted $A^\star$, and if T is a linear operator, its adjoint will be denoted $T^\star$. Moreover, as usual, p' denotes the conjugate exponent of $p \in [1,\infty)$.

We consider the analytic family $\{T_z\}_{z\in S}$ of linear operators, mapping the space of complex finite sequences into the space of measurable functions in T_Ω, as follows:

$$T_z(\{\lambda_j\}) = c_{\nu+(\nu+n/r)z}\,\mathrm{e}^{(z-\theta)^2} \sum_j \lambda_j \Delta(v_j)^{\nu+(\nu+n/r)z+n/r} \Delta^{-\nu-(\nu+n/r)z-n/r}\left(\frac{\cdot - \overline{w}_j}{\mathrm{i}}\right).$$

This analytic family of operators is also admissible.

Theorem 3.2. *Let $t \in \mathsf{R}$, $2 < q_0 < Q_\nu + 1$, $q_1 = 1$.*

(i) *For $k = 0,1$, the operator $T_{k+\mathrm{i}t}$ is bounded from $l^{q_k}_\nu$ to $A^{q_k}_\nu$. More precisely, there exists a constant M_k independent of t such that*

$$\|T_{k+\mathrm{i}t}(\{\lambda_j\})\|_{A^{q_k}_\nu} \leqslant M_k \|\{\lambda_j\}\|_{l^{q_k}_\nu}.$$

(ii) *For $\varphi \in (0,1)$, $\frac{1}{q} = 1 - \varphi + \frac{\varphi}{q_0}$, the operator T_φ is bounded from l^q_ν to A^q_ν.*

(iii) *Moreover, if δ is small enough, the operator $T_\theta : l^q_\nu \to A^q_\nu$, $\frac{1}{q} = 1 - \theta + \frac{\theta}{q_0}$, is also onto.*

Proof. (i) The case $k=1$ follows by a direct computation. Suppose next that $k=0$. An easy computation gives that for every $g \in (A^{q_0}_\nu)^\star$,

$$T^\star_{\mathrm{i}t} g = \{\mathsf{P}^\star_{\mathrm{i}t} g(w_j)\}_j. \tag{2}$$

Furthermore,

$$\mathsf{P}^{\star}_{\mathrm{i}t} g(\zeta) = \Delta(\Im\mathrm{m}\,\zeta)^{-\mathrm{i}(\nu+n/r)t} H(\zeta), \tag{3}$$

where

$$H(\zeta) = \mathrm{e}^{(-\theta-\mathrm{i}t)^2} c_{\nu-\mathrm{i}(\nu+n/r)t} \left\langle g, \Delta^{-\nu-n/r-\mathrm{i}(\nu+n/r)t}\left(\frac{\cdot - \bar{\zeta}}{\mathrm{i}}\right)\right\rangle_{(A^{q_0}_\nu)^\star, A^{q_0}_\nu}.$$

It follows that

$$\|\mathsf{P}^{\star}_{\mathrm{i}t} g\|_{L^{q_0'}_\nu} = \|H\|_{A^{q'_0}_\nu}. \tag{4}$$

Moreover, by Theorem 2.1, we get

$$\|\mathsf{P}^{\star}_{\mathrm{i}t} g\|_{L^{q_0'}_\nu} \leqslant m_0' \|g\|_{(A^{q_0}_\nu)^\star}, \tag{5}$$

where the constant m_0' does not depend on t. So the function H belongs to $A^{q'_0}_\nu$. We use the following lemma:

Lemma 3.3 (cf. [3]). *Let $p \geqslant 1$ and $f \in A^p_\nu$. There exists a constant $d_\delta > 0$ such that*

$$\left\|\left\{f(w_j)\right\}_j\right\|_{l^p_\nu} \leqslant d_\delta \|f\|_{A^p_\nu} \tag{6}$$

and if δ is small enough, we have the converse inequality:

$$\|f\|_{A^p_\nu} \leqslant 2d_\delta \left\|\left\{f(w_j)\right\}_j\right\|_{l^p_\nu}. \tag{7}$$

Combining (2), (6) and (5) in this order implies

$$\|T^\star_{\mathrm{i}t} g\|_{l^{q_0'}_\nu} = \left\|\left\{\mathsf{P}^\star_{\mathrm{i}t} g(w_j)\right\}\right\|_{l^{q_0'}_\nu} \leqslant d_\delta \|\mathsf{P}^\star_{\mathrm{i}t} g\|_{L^{q_0'}_\nu} \leqslant d_\delta m_0' \|g\|_{(A^{q_0}_\nu)^\star}.$$

It then follows that $\|T_{\mathrm{i}t}(\{\lambda_j\})\|_{A^{q_0}_\nu} \leqslant M_0 \|\{\lambda_j\}\|_{l^{q_0}_\nu}$, where M_0 is a constant independent of t.

(ii) The result follows from (i) using the interpolation of the analytic family $\{T_z\}_{z\in S}$ of operators.

(iii) If δ is small enough, then by (7), we get:

$$\|\mathsf{P}^\star_\theta g\|_{L^{q'}_\nu} \leqslant 2d_\delta \|T^\star_\theta g\|_{l^{q'}_\nu}. \tag{8}$$

By assertion (ii) of Theorem 2.1, there exists a constant a such that

$$\|g\|_{(A^q_\nu)^\star} \leqslant a \|\mathsf{P}^\star_\theta g\|_{L^{q'}_\nu}; \tag{9}$$

so, by (8) and (9), we obtain $\|g\|_{(A^q_\nu)^\star} \leqslant 2ad_\delta \|T^*_\theta g\|_{l^{q'}_\nu}$ and thus, T_θ is onto. □

4. An explicit analytic mapping on S, with values in $A^{p_0}_\nu + A^{p_1}_\nu$

Let $F(A^1_\nu, A^{p_1}_\nu)$ denote the space of all mappings $f : S \to A^1_\nu + A^{p_1}_\nu$ such that (1) f is analytic in the interior of S; (2) f is continuous and bounded on S; (3) the mappings $t \mapsto f(k+\mathrm{i}t)$, $k = 0, 1$, are continuous from R to $A^{q_k}_\nu$ with $q_0 = p_1$ and $q_1 = 1$. The space $F(A^1_\nu, A^{p_1}_\nu)$ is a Banach space with the norm $\|f\|_F := \max_{k=0,1}\{\sup_{t\in\mathsf{R}} \|f(k+\mathrm{i}t)\|_{A^{p_k}_\nu}\}$. Moreover, $[A^1_\nu, A^{p_1}_\nu]_\theta = \{g \in A^1_\nu + A^{p_1}_\nu\colon\ \exists f \in F(A^1_\nu, A^{p_0}_\nu),\ g = f(\theta)\}$ is the complex interpolation space between A^1_ν and $A^{p_1}_\nu$; this space is a Banach space under the norm $\|g\|_{[A^1_\nu, A^{p_1}_\nu]_\theta} = \|g\|_\theta = \inf\{\|f\|_F\colon\ g = f(\theta),\ f \in F(A^1_\nu, A^{p_1}_\nu)\}$. Since $[L^1_\nu, L^{p_1}_\nu]_\theta = L^p_\nu$ with equality of norms, it is easy to get that $[A^1_\nu, A^{p_1}_\nu]_\theta \subset A^p_\nu$ with bounded inclusion. To get the converse inclusion, as we said, we want to give an explicit function $f \in F(A^1_\nu, A^{p_1}_\nu)$ such that $f(\theta)$ coincides with a given function $g \in A^p_\nu$.

Theorem 4.1. *Let $0 < \theta < 1$, $2 < p_1 < Q_\nu + 1$ and $p_0 = 1$. Define p by $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$ and for every $z \in S$, write $\alpha(z) = p(1 - z + \frac{z}{p_1})$.* (1) *Then for every $\{\lambda_j\} \in l^p_\nu$, if we define*

$$\lambda_j(z) = \begin{cases} |\lambda_j|^{\alpha(z)} \frac{\lambda_j}{|\lambda_j|} & \text{if } \lambda_j \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

the analytic mapping $f(z) = T_z(\{\lambda_j(z)\})$ is an element of $F(A^1_\nu, A^{p_1}_\nu)$.

(2) *For every $g \in A^p_\nu$, there exists a sequence $\{\lambda_j\} \in l^p_\nu$ such that $g = T_\theta(\{\lambda_j\})$ if δ is small enough. Moreover, $g = f(\theta)$.*

Proof. (1) For every positive integer m, the function $f_m(z) = T_z(\{\lambda_j(z)\}^m_{j=1})$ is an element of $F(A^1_\nu, A^{p_1}_\nu)$. We point out that $\|\{\lambda_j(k + \mathrm{i}t)\}\|^{p_k}_{l^{p_k}_\nu} = \|\{\lambda_j\}\|^p_{l^p_\nu}$ and so, by assertion (i) of Theorem 3.2,

$$\begin{aligned} \|f - f_m\|_F &= \max\Big\{\sup_{t\in\mathbb{R}} \big\|T_{\mathrm{i}t}\big(\{\lambda_j(\mathrm{i}t)\}_{j\geqslant m+1}\big)\big\|_{A^{p_1}_\nu}, \sup_{t\in\mathbb{R}} \big\|T_{1+\mathrm{i}t}\big(\{\lambda_j(1+\mathrm{i}t)\}_{j\geqslant m+1}\big)\big\|_{A^1_\nu}\Big\} \\ &\leqslant M \max\big\{\big\|\{\lambda_j\}_{j\geqslant m+1}\big\|^{p/p_1}_{l^p_\nu}, \big\|\{\lambda_j\}_{j\geqslant m+1}\big\|^p_{l^p_\nu}\big\}. \end{aligned}$$

Since $\{\lambda_j\} \in l^p_\nu$, we get $\lim_{m\to\infty} \|f - f_m\|_F = 0$ and hence f belongs to the Banach space $F(A^{p_0}_\nu, A^1_\nu)$.

(2) By assertion (iii) of Theorem 3.2, there exists a sequence $\{\lambda_j\} \in l^p_\nu$ such that g admits the atomic decomposition $g = T_\theta(\{\lambda_j\})$. For this sequence $\{\lambda_j\} \in l^p_\nu$, if we define $\{\lambda_j(z)\}$ as in assertion (1), then $\{\lambda_j(\theta)\} = \{\lambda_j\}$ and hence $g = T_\theta(\{\lambda_j(\theta)\}) = f(\theta)$. □

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