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Uncertainty principle and L^p–L^q-sufficient pairs on noncompact real symmetric spaces

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Abstract

We consider a real semi-simple Lie group G with finite center and a maximal compact sub-group K of G. Let $G = K\exp(\overline{\mathfrak{a}_+})K$ be a Cartan decomposition of G. For $x \in G$ denote $\|x\|$ the norm of the $\mathfrak{a}_+$-component of x in the Cartan decomposition of G. Let $a > 0,\ b > 0$ and $1 \leqslant p, q \leqslant \infty$. In this Note we give necessary and sufficient conditions on $a,\ b$ such that for all K-bi-invariant measurable function f on G, if $\mathrm{e}^{a\|x\|^2} f \in L^p(G)$ and $\mathrm{e}^{b\|\lambda\|^2}\mathcal{F}(f) \in L^q(\mathfrak{a}_+^*)$ then $f = 0$ almost everywhere. ***To cite this article: S. Ben Farah, K. Mokni, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Principe d'incertitude et paires L^p–L^q-suffisantes sur les espaces symétriques réels non-compacts. On considère un groupe de Lie semi-simple réel G de centre fini et K un sous-groupe compact maximal de G. Soit $G = K\exp(\overline{\mathfrak{a}_+})K$ une décomposition de Cartan de G. Pour $x \in G$, on note $\|x\|$ la norme de la composante de x dans $\mathfrak{a}_+$. Soient $a > 0,\ b > 0$ et $1 \leqslant p, q \leqslant \infty$. Dans cette Note on donne une condition nécessaire et suffisante sur $a,\ b$ telle que pour toute fonction f mesurable et K-bi-invariante sur G, si $\mathrm{e}^{a\|x\|^2} f \in L^p(G)$ et $\mathrm{e}^{b\|\lambda\|^2}\mathcal{F}(f) \in L^q(\mathfrak{a}_+^*)$ alors $f = 0$ presque partout. ***Pour citer cet article : S. Ben Farah, K. Mokni, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

One of the rigorous formalization of the uncertainty principle in the classical Fourier analysis on $\mathbb{R}$, is to study L^p–L^q-sufficient pairs of positive functions in the following meaning. A pair of positive functions (φ, ψ) is said to be L^p–L^q-sufficient, whenever, for all measurable function f, the conditions $\varphi^{-1} f \in L^p(\mathbb{R})$ and $\psi^{-1}\hat{f} \in L^q(\mathbb{R})$ implies that $f = 0$ almost everywhere. When $p = q = \infty$, the pair is simply said to be sufficient as in [10] (p. 128).

This problem has been intensively studied in the literature, in many situations. For example, on $\mathbb{R}$, when $p = q = \infty$ and $(\varphi(x), \psi(\lambda)) = (\mathrm{e}^{-ax^2}, \mathrm{e}^{-b\lambda^2})$, we obtain the classical Hardy's theorem, see [9]. Cowling and

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Price [5] have proved that the pair $(\mathrm{e}^{-ax^2}, \mathrm{e}^{-b\lambda^2})$ is L^p–L^q-sufficient if and only if $ab \geqslant 1/4$ and p or q is finite. Analogous of these results have been also studied in [3,7,8,13–15].

Now we consider a real semi-simple Lie group G with finite center. Let $G = K\exp(\overline{\mathfrak{a}_+})K$ be a Cartan decomposition of G. Denote by $\|x\|$ the norm of the $\mathfrak{a}_+$-component of $x \in G$ in the Cartan decomposition. In [12], Narayanan and Ray, have proved that for all $1 \leqslant p \leqslant \infty$ and for all $1 \leqslant q < \infty$, the pair $(\mathrm{e}^{-a\|x\|^2}\varphi_0^{1-2/p}, \mathrm{e}^{-b\|\lambda\|^2})$ defined respectively on G and $\mathfrak{a}_+^*$ is L^p–L^q-sufficient if and only if $ab \geqslant 1/4$ (φ_0 is the spherical function below). In [4], we have proved, with Trimèche, that the pair $(h_a(x), \mathrm{e}^{-b\|\lambda\|^2})$ is L^p–L^q-sufficient if and only if $ab \geqslant 1/4$ and p or q is finite, where h_a is the heat kernel on G at time a. We note that this pair gives the correct decay condition to obtain the analogue of the above Cowling and Price result. In the other hand, Sitaram and Sundari in [14], and Cowling, Sitaram and Sundari in [6] have studied the pair $(\mathrm{e}^{-a\|x\|^2}, \mathrm{e}^{-b\|\lambda\|^2})$ for $p = q = \infty$.

The aim of this Note is to study the L^p–L^q-sufficiency of the same pair $(\mathrm{e}^{-a\|x\|^2}, \mathrm{e}^{-b\|\lambda\|^2})$ for all $1 \leqslant p, q \leqslant \infty$ and all $a > 0,\ b > 0$.

2. Notations

In this section we introduce some classical notations and results about semi-simple Lie groups. For details we refer to [11].

Let G be a connected, non compact real semi-simple Lie group with finite center and K a fixed maximal compact sub-group of G. Take $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$ such that $\mathfrak{k} = \mathrm{Lie}(K)$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. The associated Killing form defines a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}$. By duality, we define a scalar product on $\mathfrak{a}^*$ which can be extended to $\mathfrak{a}_{\mathbb{C}}^*$ as a hermitian product, denoted also by $\langle \cdot, \cdot \rangle$. Let $\|\cdot\|$ be the associated norm. As usual denote by W the Weyl group and Σ the set of all roots. Let Σ^+ be a fixed set of positive roots, Σ_0^+ the set of positive indivisible roots and $\mathfrak{a}_+$, $\mathfrak{a}_+^*$ the corresponding Weyl chambers respectively in $\mathfrak{a}$ and $\mathfrak{a}^*$. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$. We have the Cartan decomposition $G = K\exp(\overline{\mathfrak{a}_+})K$. For all $x \in G$, denote $|x| = \|x^+\|$ where x^+ is the $\mathfrak{a}_+$-component of x in the above decomposition. For all $x \in G$, let $H(x)$ be the unique element in $\mathfrak{a}$ such that $x \in K\exp H(x)N$. The spherical functions on G are defined by $\varphi_\lambda(x) = \int_K \mathrm{e}^{(\mathrm{i}\lambda-\rho)(H(xk))}\,\mathrm{d}k,\ x \in G,\ \lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

The spherical Fourier transform on G is defined by $\mathcal{F}(f)(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)\,\mathrm{d}x,\quad f \in \mathcal{D}^\natural(G)$. Let c be the Harish–Chandra-function defined on $\mathfrak{a}^*$. Then the inversion formula is given by $\mathcal{F}^{-1}(h)(x) = \int_{\mathfrak{a}_+^*} h(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}\,\mathrm{d}\lambda,\ h = \mathcal{F}(f)$.

3. Phragmén–Lindelöf type results

We need some complex analysis results for the proof of the main theorem of this paper.

Fix γ a positive measurable function on $]0,\infty[$. Suppose there exists integer $k > 0$ and a reals $\sigma_0 > 0$ and $\varepsilon_0 > 0$ such that

(i) $\forall r > 0,\ \forall x \geqslant \sigma_0\ \gamma(rx) \leqslant \mathrm{const.}\cdot\max(r^k, 1)\gamma(x)$.
(ii) $\forall \sigma > \sigma_0,\ \mathrm{d}_\gamma(\sigma) = \int_\sigma^{\sigma+1}\gamma(x)\,\mathrm{d}x \geqslant \varepsilon_0$.

Lemma 3.1. *Let f be an analytic even function on $\mathbb{C}$. Suppose that for $1 \leqslant q < +\infty,\ m \in \mathbb{N},\ M > 0$ and a constant $\nu > 0$ we have for all $z \in \mathbb{C}$*

$$\bigl|f(z)\bigr| \leqslant M\bigl(1+|z|\bigr)^m\,\mathrm{e}^{\nu\Re(z^2)} \quad \textit{and} \quad \int_0^\infty \bigl|f(x)\bigr|^q\gamma(x)\,\mathrm{d}x \leqslant M.$$

Then $f = 0$ on $\mathbb{C}$.

Using the Phragmén–Lindelöf principle (see [10], p. 36) we obtain the following result

Lemma 3.2. *Let f be an even analytic function on $\mathbb{C}$. Suppose that for $m \in \mathbb{N}$, $M > 0$ and $\nu > 0$ we have and all $z \in \mathbb{C}$ and all $x \in \mathbb{R}+$*

$$\left|f(z)\right| \leqslant M\left(1+|z|\right)^m \mathrm{e}^{\nu \Re(z^2)} \quad \textit{and} \quad \left|f(x)\right| \leqslant M.$$

Then $f = \text{const.}$ on $\mathbb{C}$.

4. The L^p–L^q version of Hardy's theorem

We start by the principal theorem of this Note.

Theorem 4.1. *Let $1 \leqslant p, q \leqslant \infty$ and $a > 0,\ b > 0$.*
If $1 \leqslant p \leqslant 2$ then the pair $(\mathrm{e}^{-a||x||^2}, \mathrm{e}^{-b||\lambda||^2})$ is L^p–L^q-sufficient if and only if $ab \geqslant 1/4$.
If $2 < p \leqslant \infty$ and $ab > 1/4$, then the pair $(\mathrm{e}^{-a||x||^2}, \mathrm{e}^{-b||\lambda||^2})$ is L^p–L^q-sufficient.

In Proposition 4.7, we prove that $ab > 1/4$ is necessary and suffisent, in the case $G = \mathrm{SL}(2, \mathbb{C})$.
The proof of this theorem is a consequence of the following results. For given $a > 0,\ b > 0,\ 1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$, let f be a K-bi-invariant measurable function on G such that $\|\mathrm{e}^{a\|x\|^2} f\|_{L^p(G)} < \infty$. The condition on f and properties of the spherical functions [11], imply that the definition of the Fourier transform can be extended to f, and $\mathcal{F}(f)$ is W-invariant and analytic on $\mathfrak{a}^*_{\mathbb{C}}$. Moreover it satisfies the properties given in the following lemma.

Lemma 4.2. *Let p' be the conjugate exponent of p. We have for all $\lambda = \xi + \mathrm{i}\eta \in \mathfrak{a}^* + \mathrm{i}\mathfrak{a}^*_+$*
if $1 < p \leqslant \infty$ then $|\mathcal{F}(f)(\lambda)| \leqslant \text{const.} \cdot (1 + \|\eta\|)^d\, \mathrm{e}^{(1/4a)\|\eta + \frac{2-p'}{p'}\rho\|^2}$,
if $p = 1$ then $|\mathcal{F}(f)(\lambda)| \leqslant \text{const.} \cdot \mathrm{e}^{(1/4a)\|\eta\|^2}$.

Let $\mu_1, \ldots, \mu_l$ be a basis of $\mathfrak{a}^*$ such that $\mathfrak{a}^*_+ = \sum_{i=1}^{l} \mathbb{R}^*_+ \cdot \mu_i$. Let $\Lambda_t = \mu_1 + t_2\mu_2 + \cdots + t_l\mu_l$ for all $t = (t_2, \ldots, t_l) \in \mathbb{R}^{l-1}$. The change of variable $(x_1, \ldots, x_l) = x(1, t_2, \ldots, t_l)$ and Fubini's theorem gives

Lemma 4.3. *If $1 \leqslant q < \infty$ and $\mathrm{e}^{b\|\lambda\|^2}\mathcal{F}f$ is in $L^q(\mathfrak{a}^*_+, |c(\lambda)|^{-2}\,\mathrm{d}\lambda)$ then*

$$\int_0^\infty \left|\mathrm{e}^{b|x|^2\|\Lambda_t\|^2}\mathcal{F}(f)(x\Lambda_t)\right|^q \left|c(x\Lambda_t)\right|^{-2} x^{l-1}\,\mathrm{d}x < +\infty, \tag{1}$$

for almost all $t_2 > 0, \ldots, t_l > 0$.

Proposition 4.4. *Let $1 \leqslant p, q \leqslant \infty$ and f a K-biinvariant measurable function on G such that*

$$\left\|\mathrm{e}^{a\|x\|^2} f\right\|_{L^p(G)} \leqslant M \quad \textit{and} \quad \left\|\mathrm{e}^{b\|\lambda\|^2}\mathcal{F}f\right\|_{L^q(\mathfrak{a}^*_+, |c(\lambda)|^{-2}d\lambda)} \leqslant M, \tag{2}$$

for $M > 0$, $a > 0$ and $b > 0$. If $ab > 1/4$ then $f = 0$ almost everywhere.

Proof. For $a < a' < 1/4b$ and $t_2 > 0, \ldots, t_l > 0$ let $t = (t_1, \ldots, t_l)$ and $g_{a',t} : \mathbb{C} \to \mathbb{C}$ be defined by $g_{a',t}(z) = \mathrm{e}^{(1/4a')\|\Lambda_t\|^2 z^2}\mathcal{F}(f)(z\Lambda_t)$ and $\gamma(x) = |c(x\Lambda_t)|^{-2}x^{l-1}$. For almost $t_2 > 0, \ldots, t_l > 0$ Lemma 3.1 or Lemma 3.2 gives that $g_{a',t} = 0$ on $\mathbb{C}$ then $\mathcal{F}(f) = 0$ on $\mathfrak{a}^*_{\mathbb{C}}$. Hence $f = 0$ almost everywhere.

Using similar proof we obtain the following result

Proposition 4.5. *Suppose $1 \leqslant p \leqslant 2$ and $1 \leqslant q \leqslant \infty$. Let f be a K-biinvariant measurable function on G such that*

$$\left\|e^{a|x|^2} f\right\|_{L^p(G)} \leqslant M \quad \textit{and} \quad \left\|e^{b\|\lambda\|^2} \mathcal{F} f(\lambda)\right\|_{L^q(\mathfrak{a}_+^*, |c(\lambda)|^{-2} d\lambda)} \leqslant M,$$

for $M > 0$, $a > 0$ and $b > 0$. If $ab = 1/4$ then $f = 0$ almost everywhere.

The heat kernel h_a is defined for $a > 0$ and is a positive K-bi-invariant $\mathcal{C}^\infty$-function on G. Using Anker's estimate [1,2] of h_a we obtain

Proposition 4.6. *If $ab < 1/4$ then for all $1 \leqslant p, q \leqslant \infty$ and $a < t < 1/4b$, h_t verifies*

$$\left\|e^{a\|x\|^2} h_t\right\|_{L^p(G)} < \infty \quad \textit{and} \quad \left\|e^{b\|\lambda\|^2} \mathcal{F}(h_t)(\lambda)\right\|_{L^q(\mathfrak{a}_+^*, |c(\lambda)|^{-2} d\lambda)} < \infty.$$

Now we consider the group $G = \mathrm{SL}(2, \mathbb{C})$ as a real Lie group. We take $K = \mathrm{SU}(2)$ and

$$\mathfrak{a} = \left\{ H_x = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} :\ x \in \mathbb{R} \right\}.$$

We can identify $\mathfrak{a}$ to $\mathbb{R}$ in such a way that $\|H_x\| = |x|$. For all real λ, the map $H_x \mapsto \lambda \cdot x$ gives an identification of $\mathfrak{a}^*$ with $\mathbb{R}$ such that $\|\lambda\|$ in $\mathfrak{a}^*$ is $|\lambda|$. Under these conditions we have $\varphi_\lambda(x) = \frac{2 \sin \lambda x}{\lambda \sinh 2x}$; $|c(\lambda)|^{-2} = \frac{\lambda^2}{4}$ and $\delta(x) = 4 \sinh^2 2x$.

For $\alpha \geqslant 0$, let g_α be defined on $\mathbb{R}$ by $g_\alpha(\lambda) = e^{-\lambda^2/4} \sin^4(\alpha\lambda)/\lambda^4$.

Proposition 4.7. *Let $2 < p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$. For all $0 \leqslant \alpha < \min(1/4, (p-2)/4p)$, the functions $f_\alpha = \mathcal{F}^{-1}(g_\alpha)$ verifies*

$$\left\|e^{|x|^2} f_\alpha\right\|_{L^p(G)} < \infty \quad \textit{and} \quad \left\|e^{\|\lambda\|^2/4} \mathcal{F} f_\alpha\right\|_{L^q(\mathfrak{a}_+^*, |c(\lambda)|^{-2}\, d\lambda)} < \infty.$$

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