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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Capacitary estimates of solutions of a class of nonlinear elliptic equations ☆

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Abstract

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and K a compact subset of $\partial\Omega$. Assume that $q \geqslant (N+1)/(N-1)$ and denote by U_K the maximal solution of $-\Delta u + u^q = 0$ in Ω which vanishes on $\partial\Omega \setminus K$. We obtain sharp upper and lower estimates for U_K in terms of the Bessel capacity $C_{2/q,q'}$ and prove that U_K is σ-moderate. In addition we relate the strong 'blow-up' points of U_K on $\partial\Omega$ to the 'thick' points of K in the fine topology associated with $C_{2/q,q'}$ and characterize these points by a path integral condition on U_K. ***To cite this article: M. Marcus, L. Véron, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Estimations capacitaires des solutions d'une classe d'équations elliptiques non linéaires. Soit Ω un domaine borné régulier de $\mathbb{R}^N$ and K un sous-ensemble compact de $\partial\Omega$. Supposons $q \geqslant (N+1)/(N-1)$ et soit U_K la solution maximale de $(\mathcal{E})$ $-\Delta u + u^q = 0$ dans Ω qui s'annulle sur $\partial\Omega \setminus K$. Nous obtenons des majorations et minorations précises de U_K au moyen de la capacité de Bessel $C_{2/q,q'}$ et montrons que U_K est σ-modérée. En outre nous corrélons les points d'explosion forte de U_K et les points épais de K pour la topologie fine associée à $C_{2/q,q'}$ et caractérisons ces points par une condition d'intégrale de chemin portant sur U_K. ***Pour citer cet article : M. Marcus, L. Véron, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit Ω un domaine de $\mathbb{R}^N$ de bord de classe C^2 et $q > 1$ un nombre réel. Si μ est une mesure de Radon sur $\Sigma := \partial\Omega$, le problème (1) possède une solution si et seulement si μ s'annulle sur les ensembles de capacité $C_{2/q,q'}$ nulle. Cette solution est unique et sera notée u_μ. Une solution positive $u \in C^2(\Omega)$ de l'Éq. (2) est σ-modérée si il existe une suite croissante de mesures de Radon positives μ_n sur Σ telles que la suite $\{u_{\mu_n}\}$ converge vers u. Le Gall [5], dans le cas $N = q = 2$, par des méthodes probabilistes, puis Marcus et Véron [6] dans le cas général

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$1 < q < (N+1)/(N-1)$, par des méthodes entièrement analytiques, ont montré que toutes les solutions de (2) étaient σ-modérées. Dans [9] Mselati a prouvé par une combinaison de méthodes probabilistes et analytiques que dans le cas $q = 2$, ce résultat demeurait vrai en toute dimension. Le théorème suivant étend au cas général l'étape clef de la construction de Mselati :

Théorème 2. *Soit $K \subset \Sigma$ un sous-ensemble compact et $q \geqslant (N+1)/(N-1)$. Alors la solution maximale U_K de* (2) *qui s'annulle sur $\Sigma \setminus K$ est σ-modérée.*

La fonction $\underline{U}_K = \sup\{u_\mu : \mu \in W_+^{2/q,q'}(\Sigma),\ \mu(K^c) = 0\}$ est σ-modérée par construction. Si $x \in \Omega$, on obtient des estimations précises de $U_K(x)$ et de $\underline{U}_K(x)$ en fonction de la capacité de Bessel de K et des distances $\rho(x)$ et $\rho_K(x)$ de x au bord et à K respectivement. On démontre alors que le quotient $U_K/\underline{U}_K$ est majoré dans Ω et on conclut comme dans [6]. Un point σ de K est dit épais pour la topologie fine associée à $C_{2/q,q'}$ si

$$J_q(K,\sigma) := \int_0^1 \left(\frac{C_{2/q,q'}(K \cap \overline{B}_t(\sigma))}{t^{N-1-2/(q-1)}}\right)^{q-1} \frac{\mathrm{d}t}{t} = \infty.$$

Tout point de K, à l'exception possible d'un sous-ensemble de capacité nulle est un point épais (on dit que cette propriété est vérifiée q-p.p.). Comme K est fermé, il contient ses points épais. Le théorème suivant caractérise de tels points.

Théorème 3. *Si $\sigma \in K$ est un point épais de K, alors* (13) *est satisfaite pour toute courbe $\Gamma \in \mathrm{Lip}([0,1],\ \Omega \cup \{\sigma\})$ vérifiant $\Gamma(0) = \sigma$ et $0 < |\Gamma(t) - \sigma| \leqslant a\rho(\gamma(t))$ pour un $a \geqslant 1$ et tout $t \in (0,1]$. Ainsi* (13) *est satisfaite q-p.p. dans K, et de façon évidente cette intégrale est finie partout en dehors de K.*

1. Main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary is of class C^2 and let $q > 1$. If μ is a Radon measure on $\Sigma := \partial\Omega$, the problem

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \qquad u = \mu \quad \text{on } \Sigma, \tag{1}$$

possesses a solution if and only if μ vanishes on sets of $C_{2/q,q'}$ capacity zero [2,7]. The solution is unique and will be denoted by u_μ. Following Dynkin and Kuznestov [3], a positive solution $u \in C^2(\Omega)$ of the equation

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{2}$$

is called σ-moderate if there exists an increasing sequence of positive Radon measures μ_n on Σ such that the sequence of solutions $\{u_{\mu_n}\}$ converges to u. For $N = q = 2$, Le Gall [5] proved, by probabilistic techniques, that all the positive solutions of (2) are σ-moderate. Marcus and Véron [6], employing purely analytic methods, established this result for arbitrary $N \geqslant 2$ and $1 < q < q_c = (N+1)/(N-1)$. Finally, by a combination of probabilistic and analytic tools, Mselati [9] extended Le Gall's result to $q = 2,\ N \geqslant 2$. (Note that $2 \geqslant q_c$ for $N \geqslant 3$.) In the present note we present certain capacitary estimates which provide an important tool for the extension of this result to arbitrary $q \geqslant q_c$. We apply these estimates to the study of positive solutions which blow up on a compact subset of the boundary.

In this note $C_{\alpha,p}$ $(0 < \alpha,\ 1 < p < \infty)$ denotes Bessel capacity in $\mathbb{R}^{N-1}$ or alternatively on a smooth manifold such as Σ. If $E \subset \Sigma$ then $C_{\alpha,p}(E)$ denotes the capacity of E relative to Σ and, if $\gamma > 0$, $C_{\alpha,p}(\gamma E)$ denotes the capacity of γE relative to $\gamma\Sigma$. Further we denote

$$\rho_E(x) := \mathrm{dist}(x, E), \quad \rho(x) := \rho_\Sigma(x)$$

and, for $K \in \Sigma$,

$$K_j(\xi) := \left\{x \in K\colon\ r_{j+1} \leqslant |x-\xi| \leqslant r_j\right\} \quad \forall \xi \in \Omega, \tag{3}$$

$$\widetilde{K}_j(\xi) := \left\{x \in K\colon\ |x-\xi| \leqslant r_j\right\} \quad \forall \xi \in \Omega,\ r_j = 2^{-j}. \tag{4}$$

For $\xi \in \mathbb{R}^N$ and $r > 0$ we denote by T_r^ξ the dilation mapping given by

$$T_r^\xi(x) = \frac{x-\xi}{r} + \xi \quad \forall x \in \mathbb{R}^N.$$

Since Ω is of class C^2 there exists $\beta_0 > 0$ such that for every $x \in \overline{\Omega}_{\beta_0} = \{z \in \overline{\Omega}\colon\ 0 \leqslant \rho(z) \leqslant \beta_0\}$, there exists a unique point $\sigma(x) \in \Sigma$ such that $|x - \sigma(x)| = \rho(x)$ and the mapping $x \mapsto (\rho(x), \sigma(x))$ is a C^2 diffeomorphism.

Throughout this Note we assume that $q \geqslant q_c$, K denotes a compact subset of Σ and c denotes a positive constant which depends only on q, N, Ω. The value of the constant may change from one occurrence to another. The notation $X \approx Y$ means $\frac{1}{c}X \leqslant Y \leqslant cX$ for some constant c.

The capacitary estimates are formulated in the following.

Theorem 1.1. (a) *There exists a constant c such that, for every positive solution u of* (2) *which vanishes on $K^c = \Sigma \setminus K$,*

$$u(x) \leqslant c\rho(x)\rho_K(x)^{-1-2/(q-1)} C_{2/q,q'}\big(T^x_{\rho_K(x)} K\big) \tag{5}$$

for every $x \in \Omega$ such that $\rho_K(x) \geqslant \frac{1}{4}\operatorname{diam} K$. For arbitrary $x \in \Omega$ we have

$$u(x) \leqslant c\rho(x)\left(\sum_{j=0}^{\infty} r_j^{-1-2/(q-1)} C_{2/q,q'}\big(T^x_{r_j} K_j(x)\big) + 1\right). \tag{6}$$

(b) *Put $\underline{U}_K = \sup\{u_\mu\colon\ \mu \in W_+^{2/q,q'}(\Sigma),\ \mu(K^c) = 0\}$. Then, there exists a constant c such that,*

$$\underline{U}_K(\xi) \geqslant c \sum_{j=0}^{\infty} r_j^{-2/(q-1)} C_{2/q,q'}\big(T^\xi_{r_j} K_j(\xi)\big), \tag{7}$$

for every $\xi \in \Omega$ such that $\rho_K(\xi) \leqslant 4\rho(\xi)$.

Finally,

$$\sum_{j=0}^{\infty} r_j^{-2/(q-1)} C_{2/q,q'}\big(T^\xi_{r_j} K_j(\xi)\big) \approx \sum_{j=0}^{\infty} r_j^{-2/(q-1)} C_{2/q,q'}\big(T^\xi_{r_j} \widetilde{K}_j(\xi)\big). \tag{8}$$

Remark. We note that, by [1, Section 5.2] (see in particular Corollary 5.2.3 and the first part of the proof of Theorem 5.2.1).

$$\begin{cases} C_{2/q,q'}(\gamma E) \approx c\gamma^{N-1-2/(q-1)} C_{2/q,q'}(E) & \forall \gamma > 0, \qquad \text{if } q > q_c, \\ C_{2/q,q'}(\gamma E) \leqslant c\gamma^{N-1-2/(q-1)} C_{2/q,q'}(E) & \forall \gamma \in (0,1), \quad \text{if } q = q_c. \end{cases} \tag{9}$$

Hence, if $q > q_c$, (7) is equivalent to

$$\underline{U}_K(\xi) \geqslant c \sum_{j=0}^{\infty} r_j^{1-N} C_{2/q,q'}\big(K_j(\xi)\big) \quad \forall x \in \Omega, \tag{10}$$

and similarly with respect to (6).

Applying this theorem we obtain:

Theorem 1.2. *Let U_K be the maximal solution of* (2) *which vanishes on $K^c = \Sigma \setminus K$. Then U_K is σ-moderate and $U_K = \underline{U}_K$.*

For the statement of our next result we introduce the following notation:

$$F_q(t; K, \xi) := C_{2/q,q'}\big(T_t^{\xi} K \cap \overline{B}_1(\xi)\big) = C_{2/q,q'}\big(T_t^{\xi}\big(K \cap \overline{B}_t(\xi)\big)\big) \quad \forall \xi \in \overline{\Omega}.$$

We observe that, in view of (9),

$$\begin{cases} F_q(t; K, \xi) \approx \dfrac{C_{2/q,q'}(K \cap \overline{B}_t(\xi))}{t^{N-1-2/(q-1)}} & \text{if } q > q_c,\ 0 < t, \\ F_q(t; K, \xi) \geqslant C_{2/q,q'}\big(K \cap \overline{B}_t(\xi)\big) & \text{if } q = q_c,\ 0 < t \leqslant 1. \end{cases} \tag{11}$$

We recall that a point $\sigma \in \Sigma$ is a thick point of K relative to $C_{2/q,q'}$ if

$$J_q(K, \sigma) := \int_0^1 \left(\frac{C_{2/q,q'}(K \cap \overline{B}_t(\sigma))}{t^{N-1-2/(q-1)}} \right)^{q-1} \frac{dt}{t} = \infty.$$

Every point of K, with the possible exception of a set of capacity zero, is a thick point of K (see [1, Corollary 6.3.17]). (Briefly we say that this property holds q-a.e.) In addition, since K is closed, it contains all its thick points.

Theorem 1.3. (a) *Given $a > 1$ there exists a constant $c(a) > 0$, depending also on q, N, Ω, such that, for every $\sigma \in K$,*

$$\frac{1}{c(a)} \int_s^1 t^{-2/(q-1)} F_q(t; K, \sigma) \frac{dt}{t} \leqslant U_K(x)$$
$$\leqslant c(a) \int_s^1 t^{-2/(q-1)} F_q(t; K, \sigma) \frac{dt}{t} + O(s) \quad \forall x \in \Omega\colon\ s = |x - \sigma| \leqslant a\rho(x). \tag{12}$$

(b) *If $\sigma \in K$ is a thick point of K then*

$$\int_0^1 U_K^{q-1}\big(\Gamma(t)\big) t \, dt = \infty, \tag{13}$$

for every curve $\Gamma \in \mathrm{Lip}([0,1], \Omega \cup \{\sigma\})$ such that, $\Gamma(0) = \sigma$ and $0 < |\Gamma(t) - \sigma| \leqslant a\rho(\gamma(t))$ for some $a \geqslant 1$ and every $t \in (0,1]$.

Thus (13) *holds q-a.e. in K. Obviously the integral is finite everywhere outside K.*

2. Sketch of the proof

Proof of Theorem 1.1 (sketch). (a) Let ϕ be the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ normalized by $\max \phi = 1$ and let λ be the corresponding eigenvalue.

The main ingredient in this proof is the construction of a linear lifting $R : C(\Sigma) \mapsto C^{0,1}(\overline{\Omega})$ such that R is monotone, $R(1) \equiv 1$ and R has the following additional property:

Let $\eta \in W^{2/q,q'}(\Sigma)$ *be a function with values in* $[0,1]$ *such that* $\eta \equiv 1$ *in a* neighborhood *of* K. *If* u *is a positive solution of* (2) *which vanishes on* K^c *then,*

$$\int_\Omega \big(u^q + \lambda u\big) R_{1-\eta}^{2q'} \phi \,\mathrm{d}x \leqslant C \|\eta\|_{W^{2/q,q'}}^{q'}, \tag{14}$$

where $R_{1-\eta}$ *is the lifting of* $1-\eta$.

The lifting is constructed as follows. For $\vartheta \in C(\partial\Omega)$, let H_ϑ denote the solution of

$$\frac{\partial H}{\partial \tau} = \Delta_\Sigma H \quad \text{in } \mathbb{R}_+ \times \Sigma, \qquad H(0,\cdot) = \vartheta(\cdot) \quad \text{in } \Sigma, \tag{15}$$

where Δ_Σ is the Laplace Beltrami operator on Σ. Then R_ϑ, the lifting of ϑ, is defined by

$$\begin{cases} R_\vartheta(x) = H_\vartheta\big(\phi^2(x), \sigma(x)\big) & \forall x \in \overline{\Omega}_{\beta_0}, \\ R_\vartheta \text{ is harmonic in } \Omega \setminus \Omega_{\beta_0} \quad \text{and} & R_\vartheta \in C(\Omega). \end{cases} \tag{16}$$

Now, if $\rho_K(x) \geqslant \operatorname{diam} K/4$ and $\operatorname{diam} K \leqslant 1$ we obtain the following pointwise estimate for positive solutions u vanishing on K^c:

$$u(x) \leqslant C\rho(x)\rho_K(x)^{-N} \int_\Omega \big(u^q + \lambda u\big)\phi \,\mathrm{d}x. \tag{17}$$

In addition we observe that (14) implies,

$$\int_\Omega \big(u^q + \lambda u\big)\phi \,\mathrm{d}x \leqslant C\, C_{2/q,q'}(K). \tag{18}$$

Therefore we obtain (6) for points x as above. The inequality can be extended to points arbitrarily close to K by a standard slicing method. Finally if $\operatorname{diam} K > 1$ we put $K = \bigcup_{j=1}^m K^j$ where $\operatorname{diam} K^j \leqslant 1$, K^j is compact and $m \leqslant m(\Omega)$ with $m(\Omega)$ depending only on Ω.

(b) We confine ourselves to the case $q > q_c$. In this case it is sufficient to prove (10). The proof employs an argument of Labutin [4] who established the analogue of (10) in the case that K is an interior singularity, i.e., $K \subset \Omega$. This is combined with an estimate of Marcus and Veron [8] which states that, if $\mu \in W^{-2/q,q}(\Sigma)$, then $\mathbb{P}[\mu]$ (=the Poisson potential of μ in Ω) belongs to $L^q(\Omega, \rho(x)dx)$ and

$$c_0^{-1} \|\mu\|_{W^{-2/q,q}} \leqslant \big\|\mathbb{P}[\mu]\big\|_{L^q(\Omega,\rho\,\mathrm{d}x} \leqslant c_0 \|\mu\|_{W^{-2/q,q}}. \tag{19}$$

Put $V = \mathbb{P}[\mu]$. Then, by the maximum principle,

$$u_\mu(x) \geqslant V(x) - \int_\Omega G(x,y) V^q(y)\,\mathrm{d}y, \tag{20}$$

where G is the Green kernel for $-\Delta$ in Ω. For a specific choice of the measure μ, such that $\operatorname{supp}\mu \subset K$, it can be shown that: (i) the second term on the right-hand side of (20) is controlled by V for all $x \in \Omega$ such that $\rho_K(x) \leqslant 4\rho(x)$, and (ii) V is bounded below by the right-hand side of (7). □

Proof of Theorem 1.2 (sketch). The solution $\underline{U}_K$ is σ-moderate. Therefore we only have to prove that $\underline{U}_K = U_K$. Combining the upper and lower estimates of Theorem 1.1 we find that there exists a constant c such that,

$$U_K \leqslant c\underline{U}_K \quad \text{in } \Omega' = \big\{x \in \Omega : \rho_K(x) < 4\rho(x) < \beta_0/2\big\}. \tag{21}$$

In the set $\{x \in \Omega\colon \rho_K(x) \geqslant 4\rho(x),\ \rho(x) < \beta_0/8\}$, (21) follows by an application of Hopf's lemma in conjunction with the Keller–Osserman estimate and a blow-up technique. In the remaining part of Ω the inequality follows by the maximum principle. Further, using an argument of [6], it can be shown that (21) implies the following:

If $\underline{U}_K < U_K$ *then there exists a solution* w *of* (2) *and a number* $b \in (0,1)$ *such that* $b\underline{U}_K < w < \underline{U}_K$.
This is impossible, because $\underline{U}_K$ is the smallest solution dominating $b\underline{U}_K$. □

Proof of Theorem 1.3 (sketch). Without loss of generality we may assume that $\beta_0 = 1$ and diam $K \leqslant 1/2$.

(a) Let $\sigma \in K$ and $x \in \Omega$ be as in (12). Applying the estimates of Theorem 1.1 and a lemma concerning an equivalence relation between sums and integrals we obtain

$$\begin{aligned} \frac{1}{c(a)} \int_s^1 t^{-2/(q-1)} F_q(t; K, x) \frac{\mathrm{d}t}{t} &\leqslant U_K(x) \\ &\leqslant c(a) \int_s^1 t^{-2/(q-1)} F_q(t; K, x) \frac{\mathrm{d}t}{t} + \mathrm{O}(s) \quad \forall x \in \Omega\colon\ s = |x - \sigma| \leqslant a\rho(x). \end{aligned} \tag{22}$$

By a straightforward computation, this implies (12).

(b) If $\sigma \in K$ is a thick point of K then, in view of (11),

$$\int_0^1 F_q(t; K, \sigma)^{q-1} \frac{\mathrm{d}t}{t} = \infty. \tag{23}$$

By (7), with $K_j(\xi)$ replaced by $\widetilde{K}_j(\sigma)$, or alternatively by (12), we obtain

$$c(a) F_q(2s; K, \sigma) s^{-2/(q-1)} \leqslant U_K(x_s), \tag{24}$$

for $x_s = \sigma + \mathbf{n}_\sigma$ where $\mathbf{n}_\sigma$ is the unit normal at σ pointing inwards. Hence,

$$c(a) F_q(2s; K, \sigma)^{q-1} \leqslant U_K(x_s)^{q-1} s^2.$$

This inequality and (23) imply (13) for the curve $s \mapsto x_s$. In the general case (13) is obtained by combining this result with a Harnack type inequality, for solutions of (2), in cones with vertex at the boundary. □

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