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Control, observation and polynomial decay for a coupled heat-wave system

Xu Zhang [a,b], Enrique Zuazua [b,*]

[a] *School of Mathematics, Sichuan University, Chengdu 610064, Sichuan Province, China*
[b] *Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain*

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Abstract

This Note is devoted to study the control, observation and polynomial decay of a linearized 1-d model for fluid–structure interaction, where a wave and a heat equation evolve in two bounded intervals, with natural transmission conditions at the point of interface. These conditions couple, in particular, the heat unknown with the velocity of the wave solution. The controllability and observability of the system through the wave component are derived from sidewise energy estimate and Carleman inequalities. As for the control and observation through the heat component, we need to develop first a careful spectral high frequency analysis for the underlying semigroup, which yields a new Ingahm-type inequality. It is shown that the controllable/observable subspace for both cases are quite different. Also, we obtain a sharp polynomial decay rate for the energy of smooth solutions. ***To cite this article: X. Zhang, E. Zuazua, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Contrôle, observation et décroissance polynomiale pour un système couplé ondes-chaleur. On considère un modèle couplé ondes-chaleur 1-d. L'intervalle $(-1, 1)$ est divisé en deux parties. Dans $(-1, 0)$ l'équation des ondes a lieu pour la variable z tandis que, dans $(0, 1)$, y résout l'équation de la chaleur. Au point d'interface on impose les conditions de transmission $y = z_t$ et $y_x = z_x$. Ces sont des conditions plus naturelles dans le contexte de l'interaction fluide–structure. Dans cette Note, suivant les techniques developpées dans nos travaux précédents on donne des résultats optimaux de contrôle et d'observation depuis le bord parabolique $x = 1$ et hyperbolique $x = -1$ et on montre la décroissance polynomiale des solutions régulières. ***Pour citer cet article : X. Zhang, E. Zuazua, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Dans cette Note on étudie le système 1-d ondes-chaleur :

* Corresponding author.
E-mail addresses: xu.zhang@uam.es (X. Zhang), enrique.zuazua@uam.es (E. Zuazua).

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$$
\begin{cases}
y_t - y_{xx} = 0 & \text{dans } (0,\infty)\times(0,1),\\
z_{tt} - z_{xx} = 0 & \text{dans } (0,\infty)\times(-1,0),\\
y(t,1) = z(t,-1) = 0, \quad y(t,0) = z_t(t,0), \quad y_x(t,0) = z_x(t,0), & t\in(0,\infty),\\
y(0) = y_0 & \text{dans } (0,1),\\
z(0) = z_0, \quad z_t(0) = z_1 & \text{dans } (-1,0).
\end{cases}
\tag{1}
$$

Il s'agit d'un modèle linéarisé pour l'interaction fluide–structure.

Un système semblable a été considéré dans [2] et [5]. Mais, dans [2] et [5] on avait imposé la condition de transmission $y(t,0) = z(t,0)$ au lieu de $y(t,0) = z_t(t,0)$, qui est plus naturelle dans le contexte de l'interaction fluide–structure. En effet, y représente l'analogue de la vitesse dans un fluide tandis que la vitesse de déformation de la structure est décrite par z_t ; cf. (1).

Dans cette Note on étudie les problémes de contrôle et d'observation aussi bien depuis le bord hyperbolique $x = -1$ que depuis le bord parabolique $x = 1$. On analyse aussi le taux de décroissance de l'énergie des solutions.

On adopte l'analyse développée dans [2] et [5] et on obtient des résultats analogues : (a) a hautes fréquences le spectre peut se décomposer en une partie « parabolique » et autre « hyperbolique » ; (b) les fonctions propres hyperboliques sont très concentrées dans l'intervalle $(-1,0)$ et donc le taux de décroissance des solutions n'est pas uniforme ; (c) l'énergie des solutions régulières décroit de manière polynômiale ; (d) le système est contrôlable dans l'espace d'énergie avec des contrôles H^1 agissant dans le bord hyperbolique $x = -1$; (e) le système est contrôlable depuis le bord parabolique $x = 1$ dans un espace très faible de fonctions ayant des coefficients de Fourier exponentiellement petits dans les composantes hyperboliques.

Les techniques utilisées combinent des estimations d'énergie au bord, les inégalités de Carleman, l'analyse spectrale et des inégalités généralisées d'Ingham pour les spectres mixtes parabolique-hyperbolique.

1. Introduction

In this Note, we consider first the null controllability problem of the following 1-d linearized model for fluid–structure interaction with boundary control either through the hyperbolic component:

$$
\begin{cases}
u_t - u_{xx} = 0 & \text{in } (0,T)\times(0,1),\\
v_{tt} - v_{xx} = 0 & \text{in } (0,T)\times(-1,0),\\
u(t,1) = 0, \quad v(t,-1) = g_1(t), & t\in(0,T),\\
u(t,0) = v_t(t,0), \quad u_x(t,0) = v_x(t,0), & t\in(0,T),\\
u(0) = u_0 & \text{in } (0,1),\\
v(0) = v_0, \quad v_t(0) = v_1 & \text{in } (-1,0),
\end{cases}
\tag{2}
$$

or through the parabolic one:

$$
\begin{cases}
u_t - u_{xx} = 0 & \text{in } (0,T)\times(0,1),\\
v_{tt} - v_{xx} = 0 & \text{in } (0,T)\times(-1,0),\\
u(t,1) = g_2(t), \quad v(t,-1) = 0, & t\in(0,T),\\
u(t,0) = v_t(t,0), \quad u_x(t,0) = v_x(t,0), & t\in(0,T),\\
u(0) = u_0 & \text{in } (0,1),\\
v(0) = v_0, \quad v_t(0) = v_1 & \text{in } (-1,0).
\end{cases}
\tag{3}
$$

Here $T > 0$ is a finite control time, which will be needed to be large enough for the control problems to have a positive answer. Similar null controllability problems for systems (2) and (3) with the transmission condition $u(t,0) = v_t(t,0)$ replaced by $u(t,0) = v(t,0)$ were considered in [5] and [2]. Note however that, the transmission condition considered in this paper is more natural from the modelling point of view: u may be viewed as the velocity of the linearized 1-d fluid; while v_t represents the velocity of the deformation of the structure.

In (2), $g_1(t) \in H_0^1(0,T)$ is the control acting on the system through the wave extreme $x = -1$; while the state space is the Hilbert space $\mathcal{H} \equiv L^2(0,1)\times H^1(-1,0)\times L^2(-1,0)$ with the canonical norm.

Put $H = \{(\phi, \psi, \eta) \mid \phi \in L^2(0,1),\ \psi \in H^1(-1,0)$ with $\psi(-1) = 0, \eta \in L^2(-1,0)\}$. Obviously, H is a Hilbert space with the norm $|(\phi, \psi, \eta)|_H = [|\phi|^2_{L^2(0,1)} + |\psi_x|^2_{L^2(-1,0)} + |\eta|^2_{L^2(-1,0)}]^{1/2}$. By means of the transposition method, it is easy to show that, for any $(u_0, v_0, v_1) \in H (\subsetneq \mathcal{H})$ and $g_1 \in H_0^1(0,T)$, system (2) admits a unique solution (u, v, v_t) in the class $C([0,T]; \mathcal{H})$ with $(u(T), v(T), v_t(T)) \in H$. Note that, of course, the trajectories of (2) are not in H unless $g_1 \equiv 0$ (since the second component of the element in H vanishes at $x = -1$).

In (3), $g_2(t) \in H_0^1(0,T)$ is the control acting on the system through the heat extreme $x = 1$; while the state space is H. Using again the transposition method, it is easy to show that, for any $(u_0, v_0, v_1) \in H$ and $g_2 \in H_0^1(0,T)$, system (3) admits a unique solution (u, v, v_t) in the class $C([0,T]; H)$.

Our first goal is to select a control g_1 (resp. g_2) such that the solution of (2) (resp. (3)) vanishes at time $t = T$. By a classical duality argument [1], this may be reduced to the obtention of boundary observability estimates for the following system through the wave and heat components, respectively.

$$\begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\infty) \times (0,1), \\ z_{tt} - z_{xx} = 0 & \text{in } (0,\infty) \times (-1,0), \\ y(t,1) = z(t,-1) = 0, \quad y(t,0) = z_t(t,0), \quad y_x(t,0) = z_x(t,0), & t \in (0,\infty), \\ y(0) = y_0 & \text{in } (0,1), \\ z(0) = z_0, \quad z_t(0) = z_1 & \text{in } (-1,0). \end{cases} \tag{4}$$

System (4) is well-posed in H. Moreover, the energy of system (4),

$$E(t) \triangleq \frac{1}{2}\left[\int_{-1}^{0} \left(\left|z_x(t,x)\right|^2 + \left|z_t(t,x)\right|^2\right) \mathrm{d}x + \int_0^1 \left|y(t,x)\right|^2 \mathrm{d}x\right],$$

decreases along trajectories. More precisely,

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\frac{1}{2} \int_0^1 |y_x|^2 \,\mathrm{d}x.$$

This formula shows that the only dissipation mechanism of system (4) comes from the heat component. The decay rate of $E(t)$ will also be addressed in this Note. As we shall see, unlike the pure heat equation or the 1-d wave equation dissipated on a subinterval, this dissipation mechanism is not strong enough to produce an exponential decay of the energy.

In order to show the boundary observability of (4) in H through the wave component, we proceed as in [5] by combining the sidewise energy estimate for the wave equation and the Carleman inequalities for the heat equation. However, due to the new transmission condition $y(t,0) = z_t(t,0)$ on the interface, some undesired lower order term occurs in the observability inequality. Hence, we will need to use the classical Compactness–Uniqueness Argument [4] to absorb it (note that this argument is not necessary in [5] and [2]). On the other hand, the functional setting of the observability inequality differs from that in [5].

As for the boundary observability estimates for (4) through the heat component, similar to [2], we need to develop first a careful spectral analysis for the underlying semigroup of (4). Our spectral analysis yields:

(a) Lack of observability of system (4) in H from the heat extreme $x = 1$ with a defect of infinite order;
(b) A new Ingham-type inequality for mixed parabolic and hyperbolic spectra;
(c) The observability of system (4) in a Hilbert space with, roughly speaking, exponentially small weight for the Fourier coefficients of the hyperbolic eigenvectors;
(d) And then the null controllability of system (3) in a Hilbert space with, roughly speaking, exponentially large weight for the Fourier coefficients of the hyperbolic eigenvectors.

We refer to [3] for detailed proof of the results in this paper and other results in this context.

2. Boundary control and observation through the wave component

We begin with the following observability estimate:

Theorem 2.1. *Let $T > 2$. Then there is a constant $C > 0$ such that every solution of Eq.* (4) *satisfies*

$$\left|\big(y(T), z(T), z_t(T)\big)\right|_H^2 \leqslant C\left|z_x(\cdot, -1)\right|_{L^2(0,T)}^2, \quad \forall (y_0, z_0, z_1) \in H. \tag{5}$$

By means of the duality argument, Theorem 2.1 yields the null controllability of (2) but with the trajectories in a Hilbert space larger than $\mathcal{H}$. In order to obtain the null controllability of (2) in $\mathcal{H}$, we need to derive another observability inequality, which reads:

Theorem 2.2. *Let $T > 2$. Then there is a constant $C > 0$ such that every solution of Eq.* (4) *satisfies*

$$\left|\big(y(T), z(T), z_t(T)\big)\right|_H^2 \leqslant C\left|z_x(\cdot, -1) - \frac{1}{T}\int_0^T z_x(t, -1)\,\mathrm{d}t\right|_{L^2(0,T)}^2, \quad \forall (y_0, z_0, z_1) \in H. \tag{6}$$

Note that Theorem 2.1 will play a key role in Section 4 when deducing the Ingham-type inequality. Theorem 2.2 states that the observability is still true by making weaker, zero average, boundary measurements. As far as we know, the fact this inequality holds is also new in the case of a simple wave equation.

As we mentioned before, similar to [5], the proof of Theorems 2.1 and 2.2 is based on the sidewise energy estimate for the wave equation and the Carleman inequalities for the heat equation. However, some elementary but key technique of lifting the underlying Hilbert space and the classical Compactness–Uniqueness Argument (see [4]) are also necessary in the proof. Note that one does need the later two techniques in [5] and [2].

Theorem 2.2 implies the null controllability of system (2) through the wave component:

Theorem 2.3. *Let $T > 2$. Then for every $(u_0, v_0, v_1) \in H$, there exists a control $g_1 \in H_0^1(0, T)$ such that the solution (u, v, v_t) of system* (2) *satisfies $u(T) = 0$ in $(0, 1)$ and $v(T) = v_t(T) = 0$ in $(-1, 0)$.*

3. Spectral analysis

System (4) can be written in an abstract form $Y_t = \mathcal{A}Y$ with $Y(0) = Y_0$. Here $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is an unbounded operator defined as follows: $\mathcal{A}Y = (f_{xx}, h, g_{xx})$, where $Y = (f, g, h) \in D(\mathcal{A})$, and $D(\mathcal{A}) \equiv \{(f, g, h) \mid f \in H^2(0, 1),\ g \in H^2(-1, 0),\ h \in H^1(-1, 0),\ f(1) = g(-1) = h(-1) = 0,\ f(0) = h(0),\ f_x(0) = g_x(0)\}$. It is easy to show that $\mathcal{A}$ generates a contractive C_0-semigroup in H with compact resolvent. Hence $\mathcal{A}$ has a sequence of eigenvalues (in $\mathbb{C}$) tending to ∞.

The main result in this section can be written:

Theorem 3.1. *The large eigenvalues of $\mathcal{A}$ can be divided into two classes $\{\lambda_\ell^p\}_{\ell=\ell_1}^\infty$ and $\{\lambda_k^h\}_{|k|=k_1}^\infty$, where ℓ_1 and k_1 are suitable positive integers, which satisfy the following asymptotic estimates as ℓ and k tend to ∞ respectively*:

$$\lambda_\ell^p = -\left(\frac{1}{2} + \ell\right)^2 \pi^2 + 2 + \mathrm{O}\big(\ell^{-1}\big), \qquad \lambda_k^h = -\frac{1}{\sqrt{2|k|\pi}} + k\pi\mathrm{i} + \frac{\mathrm{sgn}(k)}{\sqrt{2|k|\pi}}\mathrm{i} + \mathrm{O}\big(|k|^{-1}\big). \tag{7}$$

Furthermore there exist integers $n_0 > 0$, $\tilde{\ell}_1 \geqslant \ell_1$ and $\tilde{k}_1 \geqslant k_1$ such that $\{u_{j,0}, \dots, u_{j,m_j-1}\}_{j=1}^{n_0} \cup \{u_\ell^p\}_{\ell=\tilde{\ell}_1}^\infty \cup \{u_k^h\}_{|k|=\tilde{k}_1}^\infty$ form a Riesz basis of H, where $u_{j,0}$ is an eigenvector of $\mathcal{A}$ with respect to some eigenvalue μ_j with algebraic multiplicity m_j, and $\{u_{j,1}, \dots, u_{j,m_j-1}\}$ is the associated Jordan chain, and u_ℓ^p and u_k^h are eigenvectors of $\mathcal{A}$ with respect to eigenvalues λ_ℓ^p and λ_k^h, respectively.

Here and in the sequel the superindex p stands for "parabolic" while h for "hyperbolic". This theorem indeed shows that there are two distinguished branches of the spectrum at high frequencies. The parabolic eigenvalues are indeed close to those of a heat equation while the hyperbolic ones behaves like those of the wave equation with a weak damping term. It can be shown that the first order approximation of the parabolic component of the parabolic eigenvalues are eigenfunctions of the heat equation in the interval $(0,1)$ with Dirichlet boundary condition at $x=1$ and Neumann boundary condition at the transmission point $x=0$; while the first order approximation of the hyperbolic ones are eigenfunctions of the wave equation in the interval $(-1,0)$ with Dirichlet boundary conditions. The leading terms of the parabolic and hyperbolic eigenvalues in (7) correspond to the same boundary conditions. Note that the first order approximation of eigenvectors for the system discussed in [2] have a different behavior since the boundary conditions for parabolic and hyperbolic eigenvectors are reversed in that case.

4. Ingham-type inequality for mixed parabolic-hyperbolic spectra

By means of our spectral decomposition result the observability estimate (5) can be written as an Ingham-type inequality (recall Theorem 3.1 for $n_0,\ m_j,\ \tilde{\ell}_1, \tilde{k}_1$ and $\mu_j,\ \lambda_\ell^p$ and λ_k^h):

Lemma 4.1. *Let $T>2$. Then there is a constant $C=C(T)>0$ such that*

$$\sum_{j=1}^{n_0}\sum_{k=0}^{m_j-1}|a_{j,k}|^2+\sum_{\ell=\tilde{\ell}_1}^{\infty}|a_\ell|^2\,\mathrm{e}^{2(T-1)\operatorname{Re}\lambda_\ell^p}+\sum_{|k|=\tilde{k}_1}^{\infty}|b_k|^2$$
$$\leqslant C\int_0^T\left|\sum_{j=1}^{n_0}\mathrm{e}^{\mu_j t}\sum_{k=0}^{m_j-1}a_{j,k}t^k+\sum_{\ell=\tilde{\ell}_1}^{\infty}a_\ell\,\mathrm{e}^{\lambda_\ell^p t}+\sum_{|k|=\tilde{k}_1}^{\infty}b_k\,\mathrm{e}^{\lambda_k^h t}\right|^2\mathrm{d}t \tag{8}$$

holds for all complex numbers $a_{j,k}$ $(k=0,1,\ldots,m_j-1;\ \ j=1,2,\ldots,n_0)$, and all square-summable sequences $\{a_\ell\}_{\ell=\tilde{\ell}_1}^{\infty}$ and $\{b_k\}_{|k|=\tilde{k}_1}^{\infty}$ in $\mathbb{C}$.

The Ingham-type inequality (8) is similar to the one in [2] but for different sequences $\{\lambda_\ell^p\}_{\ell=\tilde{\ell}_1}^{\infty}$ and $\{\lambda_k^h\}_{|k|=\tilde{k}_1}^{\infty}$. At this point we would like to underline that, as far as we know, there is no a direct proof of inequalities of the form (8) in the literature devoted to this issue. It is in fact a consequence of estimate (5) obtained by PDE techniques and the spectral analysis above.

5. Boundary control and observation through the heat component

We begin with the following negative result on the observability for system (4) in H, which implies the lack of boundary observability in H from the heat component with a defect of infinite order.

Theorem 5.1. *Let $T>0$ and $s\geqslant 0$. Then* $\sup_{(y_0,z_0,z_1)\in H\setminus\{0\}}\frac{|(y(T),z(T),z_t(T))|_H}{|y_x(\cdot,1)|_{H^s(0,T)}}=+\infty$, *where (y,z,z_t) is the solution of system* (4) *with initial data* (y_0,z_0,z_1).

Theorem 5.1 is a consequence of Theorem 3.1. Indeed, from Theorem 3.1, one may deduce that the parabolic component of solutions of system (4) decays rapidly while its hyperbolic component is "almost" conservative. Moreover, the hyperbolic eigenvectors are mostly concentrated on the wave interval. This makes the observability inequality from the heat extreme to fail in any Sobolev space.

By means of the well-known duality relationship between controllability and observability, from Theorem 5.1, one concludes that system (3) is not null controllable in H with $L^2(0,T)$-controls at $x=1$ neither, with controls in any negative index Sobolev space of the form $H^{-s}(0,T)$.

However, the Ingham-type inequality (8), combined with Theorem 3.1 and a sharp description of the asymptotic form of eigenvectors, allows one to get an observability inequality from the parabolic extreme in space with suitable exponential weights in the Fourier coefficients. This is precisely what we shall do in the sequel.

Put (recall Theorem 3.1 for n_0, m_j, $u_{j,k}$, $\tilde{\ell}_1$, $\tilde{k}_1$, u_ℓ^p and u_k^h)

$$V = \left\{ \sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} a_{j,k} u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell u_\ell^p + \sum_{|k|=\tilde{k}_1}^{\infty} b_k u_k^h \;\middle|\; a_{j,k}, a_\ell, b_k \in \mathbb{C},\ \sum_{\ell=\tilde{\ell}_1}^{\infty} |a_\ell|^2 + \sum_{|k|=\tilde{k}_1}^{\infty} |k| \mathrm{e}^{\sqrt{2|k|\pi}} |b_k|^2 < \infty \right\},$$

$$V' = \left\{ \sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} a_{j,k} u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell u_\ell^p + \sum_{|k|=\tilde{k}_1}^{\infty} b_k u_k^h \;\middle|\; a_{j,k}, a_\ell, b_k \in \mathbb{C},\ \sum_{\ell=\tilde{\ell}_1}^{\infty} |a_\ell|^2 + \sum_{|k|=\tilde{k}_1}^{\infty} \frac{|b_k|^2}{|k| \mathrm{e}^{\sqrt{2|k|\pi}}} < \infty \right\}.$$

V and V', endowed with their canonical norms, are mutually dual Hilbert spaces.

We have the following null controllability result on system (3):

Theorem 5.2. *Let $T > 2$. Then for every $(u_0, v_0, v_1) \in V$, there exists a control $g_2 \in H_0^1(0, T)$ such that the solution (u, v, v_t) of system* (3) *satisfies $u(T) = 0$ in $(0, 1)$ and $v(T) = v_t(T) = 0$ in $(-1, 0)$.*

In order to prove Theorem 5.2, we need to derive the following key observability estimate:

Theorem 5.3. *For any $T > 2$, there is a constant $C > 0$ such that every solution of* (4) *satisfies*

$$\big|\big(y(T), z(T), z_t(T)\big)\big|_{V'}^2 \leqslant C \big|y_x(\cdot, 1)\big|_{L^2(0,T)}^2, \quad \forall (y_0, z_0, z_1) \in V'. \tag{9}$$

Inequality (9) follows from Lemma 4.1 together with Theorem 3.1.

6. Polynomial decay rate

According to the asymptotic form of the hyperbolic eigenvalues in (7) it is clear that the decay rate of the energy is not uniform. Indeed, as (7) shows, $\operatorname{Re} \lambda_k^h \sim -c/\sqrt{|k|}$ for a positive constant $c > 0$. In this situation, the best we can expect is a polynomial decay rate for sufficiently smooth solutions. The following result is a consequence of Theorem 3.1, which provides a sharp polynomial decay rate.

Theorem 6.1. *There is a constant $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A})$, the solution of* (4) *satisfies $|(y(t), z(t), z_t(t))|_H \leqslant Ct^{-2} |(y_0, z_0, z_1)|_{D(\mathcal{A})}$, $\forall t > 0$.*

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References

[1] J.-L. Lions, Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués. Tome 1 : Contrôlabilité exacte, in: RMA, Vol. 8, Masson, Paris, 1988.

[2] X. Zhang, E. Zuazua, Polynomial decay and control of a 1-d model for fluid–structure interaction, C. R. Acad. Sci. Paris, Ser. I, to appear.

[3] X. Zhang, E. Zuazua, Polynomial decay and control of a hyperbolic-parabolic coupled system, Preprint.

[4] E. Zuazua, Contrôlabilité exacte en un temps arbitrairement petit de quelques modèles de plaques, Appendix I in [1], 465–491.

[5] E. Zuazua, Null control of a 1-d model of mixed hyperbolic-parabolic type, in: J.L. Menaldi, et al. (Eds.), Optimal Control and Partial Differential Equations, IOS Press, 2001, pp. 198–210.