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Volume-convergent sequences of Haken 3-manifolds

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Abstract

Let M be a closed orientable 3-manifold and let $\mathrm{Vol}(M)$ denote its Gromov simplicial volume. This paper is devoted to the study of sequences of non-zero degree maps $f_i : M \to N_i$ to Haken manifolds. We prove that any sequence of Haken manifolds (N_i, f_i), satisfying $\lim_{i\to\infty} \deg(f_i) \times \mathrm{Vol}(N_i) = \mathrm{Vol}(M)$ is finite up to homeomorphism. As an application, we deduce from this fact that any closed orientable 3-manifold with zero Gromov simplicial volume and in particular any graph manifold dominates at most finitely many Haken 3-manifolds. ***To cite this article: P. Derbez, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Suites de variétés Haken dont les volumes convergent. Soit M une 3-variété close orientable et désignons par $\mathrm{Vol}(M)$ le volume simplicial de Gromov de M. Cette Note est consacrée à l'étude des applications de degré non-nul $f_i : M \to N_i$ où chaque N_i est une variété Haken. Le résultat principal affirme que toute suite (N_i, f_i) de variétés Haken satisfaisant $\lim_{i\to\infty} \deg(f_i) \times \mathrm{Vol}(N_i) = \mathrm{Vol}(M)$ est finie, à homéomorphisme près. Ce résultat implique en particulier que toute 3-variété close orientable dont le volume simplicial de Gromov est nul (en particulier toute variété graphée) domine au plus un nombre fini de variétés Haken. ***Pour citer cet article : P. Derbez, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Dans cette Note on s'intéresse à la question suivante qui apparaît dans la liste de problèmes de Kirby [6, 3.100].

Question. Soit M une 3-variété orientable. Y-a-t-il au plus un nombre fini de variétés irréductibles closes telles qu'il existe une application de degré non-nul $f : M \to N$?

On adopte ici la définition suivante qui apparaît dans [1] : on dit qu'une 3-variété compacte orientable M domine une variété N si il existe une application propre de degré non nul $f : M \to N$. En particulier, on dit que M 1-*domine* N lorsque f est de degré un. Il y a plusieurs réponses partielles à cette question quand les variétés au but sont *géometriques*. Rappelons qu'une 3-variété close est dite *géometrique* si elle admet l'une

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des géométries suivantes : H^3 (hyperbolique), $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, $H^2 \times E^1$, *Sol* (la géométrie du groupe de Lie obtenu par produit semi-direct de $\mathbf{R}$ par $\mathbf{R}^2$, où $\mathbf{R}$ agit sur $\mathbf{R}^2$ par $(t,(x,y)) \to (\mathrm{e}^t x, \mathrm{e}^{-t} y)$), *Nil*, E^3 (euclidienne), $S^2 \times E^1$, S^3 (sphérique). Dans [9], Soma montre que toute 3-variété orientable close domine au plus un nombre fini de variétés hyperboliques. Dans [3], Hayat-Legrand, Wang et Zieschang montrent que toute 3-variété close orientable 1-domine au plus un nombre fini de variétés sphériques et dans [12] Wang et Zhou prouvent que toute 3-variété orientable close 1-domine au plus un nombre fini de variétés admettant l'une des géométries suivantes : $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, $H^2 \times E^1$, *Sol*, *Nil*. On s'intéresse ici à la question précédente lorsque les variétés au but sont Haken (une 3-variété close est Haken si elle est orientable, irréductible et si elle contient une surface fermée orientable $F \not\simeq \mathbf{S}^2$ incompressible).

Soit M une 3-variété orientable close et soit $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ une suite de 3-variétés dominées par M via des applications de degré non nul $f_i : M \to N_i$. Rappelons qu'étant donnée une variété close M on peut lui associer un nombre réel noté $\mathrm{Vol}(M)$ appelé le volume simplicial de Gromov de M (voir [2]). La définition du volume simplicial de Gromov implique immédiatement que : $\sup_{i\in\mathbf{N}}(\deg(f_i) \times \mathrm{Vol}(N_i)) \leqslant \mathrm{Vol}(M)$. On dira alors qu'une suite de 3-variétés closes orientables $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ constitue *une suite de variétés dominées par M dont les volumes convergent* si pour chaque i il existe une application $f_i : M \to N_i$ de degré non nul vérifiant la condition suivante : $\lim_{i\to\infty}(\deg(f_i) \times \mathrm{Vol}(N_i)) = \mathrm{Vol}(M)$. En particulier une suite de 3-variétés 1-dominées par M dont les volumes convergent satisfait, comme $\deg(f_i) = 1$ pour tout $i \in \mathbf{N}$, $\lim_{i\to\infty} \mathrm{Vol}(N_i) = \mathrm{Vol}(M)$. Dans cette Note, on donne une réponse affirmative à la question précédente dans le cas de suites de variétés Haken dont les volumes convergent.

Théorème 1. *Soit M une* 3*-variété orientable close. Toute suite $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ de variétés non-géométriques Haken dominées par M dont les volumes convergent est finie.*

En combinant ce résultat avec [12, Corollary 1], [3] et [9, Theorem 1] on établit immédiatement :

Corollaire 2. *Soit M une* 3*-variété close orientable. Toute suite $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ de variétés Haken ou géométriques* 1*-dominées par M dont les volumes convergent est finie.*

Remarque. Notons que la conjecture de géométrisation de Thurston implique qu'une variété de dimension trois irréductible doit être Haken ou géométrique.

Quand $\mathrm{Vol}(M) \neq 0$ les éléments d'une suite $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ de 3-variétés dominées par M dont les volumes convergent n'admettent aucune des géométries suivantes : $\mathbf{H}^2 \times E^1$, *Sol*, *Nil*, $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, $\mathbf{S}^3$, E^3, $\mathbf{S}^2 \times E^1$ pour i suffisamment grand. Lorsque $\mathrm{Vol}(M) = 0$ alors toute suite $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ de 3-variétés dominées par M est convergente en volume et $\mathrm{Vol}(N_i) = 0$ pour tout i. En particulier les N_i ne sont jamais des 3-variétés hyperboliques. Comme toute variété de volume simplicial nul qui domine une variété Haken ou géométrique est graphée, on obtient donc :

Corollaire 3. (i) *Toute* 3*-variété close orientable M dont le volume simplicial de Gromov est non nul domine au plus un nombre fini de variétés Haken ou géométriques de même volume simplicial que M,*

(ii) *Toute* 3*-variété graphée close domine* (*resp.* 1*-domine*) *au plus un nombre fini de variétés non-géométriques Haken* (*resp. Haken ou géométriques*).

Notons que les variétés considérées ici sont supposées closes uniquement par simplicité. Les résultats précédents restent vrais si on remplace « 3-variété close » par « 3-variété compacte à bord non vide » et « application de degré un ou non nul » par « application propre de degré non nul » (une application $f : M \to N$ est propre si $f^{-1}(\partial N) = \partial M$).

1. Definitions and results

The main purpose of this paper is to deal with the following very natural question which appears in the Kirby's Problem List [6, 3.100].

Question. Let M be a closed orientable 3-manifold. Are there at most finitely many closed irreducible and orientable 3-manifolds such that there exists a non-zero degree map $f : M \to N$?

According to [1] we say that a compact orientable 3-manifold M dominates an other one N if there is a proper non-zero degree map $f : M \to N$. In particular, we say that M 1-*dominates* N when f is a degree one map. There are many partial results on this question obtained recently when the targets are assumed *geometric*. Recall that a closed 3-manifold is called *geometric* if it admits one of the following geometries: H^3 (hyperbolic), $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, $H^2 \times E^1$, *Sol* (the geometry of the Lie group $\mathbf{R}$ semidirect product with $\mathbf{R}^2$, where $\mathbf{R}$ acts on $\mathbf{R}^2$ by $(t,(x,y)) \to (\mathrm{e}^t x, \mathrm{e}^{-t} y)$), *Nil*, E^3 (Euclidian), $S^2 \times E^1$, S^3 (spherical). In [9], Soma shows that any orientable closed 3-manifold dominates finitely many hyperbolic 3-manifolds. In [3], Hayat-Legrand, Wang and Zieschang prove that any orientable closed 3-manifold 1-dominates finitely many spherical 3-manifolds and in [12] Wang and Zhou show that any orientable closed 3-manifold 1-dominates finitely many 3-manifolds supporting one of the following geometry: $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, $H^2 \times E^1$, *Sol*, *Nil*. We deal here with the above question when the targets are Haken manifolds (a closed 3-manifold is Haken if it is orientable irreducible and if it contains an incompressible closed orientable surface $F \not\simeq \mathbf{S}^2$).

Let M be a closed orientable 3-manifold and let $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ be a sequence of 3-manifolds dominated by M via non-zero degree maps $f_i : M \to N_i$. Recall that given a closed orientable 3-manifold M one can associate to M a real number $\mathrm{Vol}(M)$ so-called the Gromov simplicial volume of M (see [2]). It follows from the definition of the Gromov simplicial volume that $\sup_{i\in\mathbf{N}}(\deg(f_i) \times \mathrm{Vol}(N_i)) \leqslant \mathrm{Vol}(M)$. We say that a sequence of closed orientable 3-manifolds $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ is a *volume-convergent sequence* dominated by M if for any i there exists a non-zero degree map $f_i : M \to N_i$ satisfying $\lim_{i\to\infty}(\deg(f_i) \times \mathrm{Vol}(N_i)) = \mathrm{Vol}(M)$. In particular a volume-convergent sequence of 3-manifolds 1-dominated by M always satisfies $\lim_{i\to\infty} \mathrm{Vol}(N_i) = \mathrm{Vol}(M)$, since $\deg(f_i) = 1$ for any $i \in \mathbf{N}$. The main purpose of this article is to give an affirmative answer to the above question for any volume-convergent sequence of Haken 3-manifolds.

Theorem 1.1. *Let M be a closed orientable 3-manifold. Then any volume-convergent sequence $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ of non-geometric Haken 3-manifolds dominated by M is finite.*

Then combining this result with [12, Corollary 1], [3] and [9, Theorem 1] we get the following corollary:

Corollary 1.2. *Let M be a closed orientable 3-manifold. Then any volume-convergent sequence $(N_i)_{i\in\mathbf{N}}$ of Haken or geometric closed 3-manifolds 1-dominated by M is finite.*

Remark. Note that the geometrization conjecture of Thurston implies that any irreducible 3-manifold is either Haken or geometric.

When $\mathrm{Vol}(M) \neq 0$ then the elements of a volume-convergent sequence $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ of 3-manifolds dominated by M do not admit the following geometries: $\mathbf{H}^2 \times E^1$, *Sol*, *Nil*, $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, $\mathbf{S}^3$, E^3, $\mathbf{S}^2 \times E^1$ for i sufficiently large. When $\mathrm{Vol}(M) = 0$ then any sequence $\mathcal{N} = (N_i)_{i\in\mathbf{N}}$ of 3-manifolds dominated by M is volume-convergent and $\mathrm{Vol}(N_i) = 0$ for any i. In particular the N_i are not hyperbolic manifolds. Thus since any 3-manifold with zero Gromov simplicial volume dominating a Haken or geometric 3-manifold is a graph manifold, we get the following:

Corollary 1.3. (i) *Any closed orientable* 3*-manifold M with non-trivial Gromov simplicial volume dominates at most finitely many Haken or geometric* 3*-manifolds with the same Gromov simplicial volume as M,*

(ii) *Any closed graph manifold dominates* (*resp.* 1*-dominates*) *at most finitely many non-geometric Haken* (*resp. Haken or geometric*) 3*-manifolds.*

Note that the given 3-manifolds are assumed closed only by simplicity. The above results remain true when we replace "closed 3-manifold" by "compact 3-manifold with non-empty boundary" and "non-zero or degree one map" by "proper non-zero degree map" (a map $f : M \to N$ is proper if $f^{-1}(\partial N) = \partial M$).

2. On the proof of Theorem 1.2

First recall that given a closed Haken manifold N we denote by $\mathcal{T}_N$ the Jaco–Shalen–Johannson family of tori of N and by $\mathcal{S}(N)$, resp. $\mathcal{H}(N)$, the union of the Seifert fibered manifolds, resp. hyperbolic manifolds, of $N \setminus \mathcal{T}_N$ (see [4, Splitting Theorem], [5] and [11, Uniformization Theorem]). In the following the family $\mathcal{T}_N$ will be also called the *canonical* or *characteristic* family of tori of N. The following definition will be convenient. Let N be a closed Haken manifold and let $(\mathcal{H}(N), \mathcal{S}(N), \mathcal{T}_N)$ denote its geometric decomposition. We say that a co-dimension zero subset Q of N is a graph submanifold if Q is a submanifold (not necessarily connected) of $N \setminus \mathcal{H}(N)$ whose boundary, if non-empty, is made of some components of $\partial\mathcal{S}(N) \subset \mathcal{T}_N$.

The idea of the proof of Theorem 1.1 is to use the geometrical decomposition for Haken manifold to have a reduction to the case of domination of geometric 3-manifolds. Since the target manifolds are irreducible then using the Milnor decomposition Theorem for compact 3-manifolds [7] one can assume, possibily passing to a subsequence, that M is an irreducible 3-manifold distinct from $S^1 \times S^2$. The strategy consists in proving the following point:

($*$) there exists a finite collection of 3-manifolds $\mathcal{G}$ depending only on M such that any component Q of $N_i \setminus \mathcal{T}_{N_i}$, $i \in \mathbf{N}$, is properly dominated by at least one element of $\mathcal{G}$.

Then one can use the known results to the above question for geometric 3-manifolds. In order to prove this point we first state the following definition for convenience. Let M be an orientable irreducible closed 3-manifold and let S be a Seifert or a hyperbolic 3-manifold whose boundary, if non-empty, consists of tori. We say that S is *locally dominated by M* if there exists a closed Haken manifold N (which is not a *Sol*-manifold) such that S is homeomorphic to a component of $N - \mathcal{T}_N$ and if there is a non-zero degree map $f : M \to N$. If, in addition, $\mathrm{Vol}(M) = \deg(f)\,\mathrm{Vol}(N)$, we say that S is *locally well-dominated* by M.

When Q is a hyperbolic 3-manifold point ($*$) comes from the following result of Soma (see [10, Key Lemma]).

Lemma 2.1. *Any closed connected irreducible* 3*-manifold M admits a finite set $\mathcal{G} = \{G_1, \ldots, G_n\}$ of complete metric spaces obtained from finitely many non-degenerate oriented* 3*-simplices in $\mathbf{H}^3$ by identifying faces of these simplices by orientation-reversing isometries such that, the interior of any compact* 3*-manifold endowed with a complete finite volume hyperbolic structure locally dominated by M is properly dominated by at least one G_i of $\mathcal{G}$.*

Here we have to prove the analogous result for Seifert fibered 3-manifolds locally well-dominated by M. More precisely Lemma 2.1 implies that if $(N_i)_{i\in\mathbf{N}}$ is a volume-convergent sequence of Haken manifolds dominated by M via maps $f_i : M \to N_i$ then, possibly passing to a subsequence, $\mathcal{H}(N_i) \simeq \mathcal{H}(N_j)$ for any $i, j \in \mathbf{N}$ and $\deg(f_i)$ is a constant denoted by $p > 0$. Hence it is sufficient to prove point ($*$) when Q is a Seifert fibered manifold and the maps $f_i : M \to N_i$ satisfy $\mathrm{Vol}(M) = \deg(f_i)\mathrm{Vol}(N_i)$. Thus our key lemma states as follows.

Key Lemma. *For any orientable closed irreducible* 3*-manifold M, there exists a finite collection $\mathcal{G} = \{G_1, \ldots, G_n\}$ of* 3*-manifolds such that any Seifert fibered manifold S locally well-dominated by M is* (*properly*) *dominated by at least one G_i of $\mathcal{G}$.*

Note that Soma's proof of Lemma 2.1 uses the geometry of the hyperbolic space and in particular the isotropy in hyperbolic geometry is crucial for "locally hyperbolizing" certain simplicial subcomplexes of M. This method cannot be adapted in the Seifert case since the geometry is not isotropic (indeed there is an invariant direction corresponding to the Seifert fibration).

The proof of the Key Lemma is based on the observation that when $\mathrm{Vol}(M) = \deg(f)\,\mathrm{Vol}(N)$ then we can "control" in a certain sense the "essential part" of $f^{-1}(\mathcal{T}_N)$. Actually one can show, up to homotopy, that this essential part is in $\mathcal{T}_M$ which is crucial in our proof since this ensures that the genus of the essential components of $f^{-1}(\mathcal{T}_N)$ is bounded independently on N. This control cannot be accomplished when $\mathrm{Vol}(M) > \deg(f)\,\mathrm{Vol}(N)$. Indeed, consider for example a degree one map from a closed hyperbolic 3-manifold M to a graph manifold N. Such a map can be built by taking a hyperbolic nul-homotopic knot k in a graph manifold N and by gluing a solid torus along $\partial(N \setminus k)$ in such a way that the resulting manifold M is hyperbolic, then the degree of the obvious decomposition map $f : M \to N$ is one. In this case one can clearly not control the genus of the components of $f^{-1}(\mathcal{T}_N)$.

3. On the proof of the Key Lemma

Let Q be a compact oriented three manifold whose boundary is made of tori $T_1, \ldots, T_k$. For each $i = 1, \ldots, k$ we fix generators $l_i,\ m_i$ of $\pi_1 T_i$. Let $\mathcal{P}_* = \{(p,q) \in \mathbf{Z} \times \mathbf{Z}, (p,q) = 1\} \cup \{\infty\}$. If $d_1, \ldots, d_k$ are in $\mathcal{P}_*$ we denote by $Q_{d_1,\ldots,d_k}$ the 3-manifold obtained from Q by gluing to each $T_i,\ i = 1, \ldots, k$, a solid torus $S^1 \times D^2$ identifying a meridian $m = \{z_0\} \times \partial D^2$ with $p_i l_i + q_i m_i$ when $d_i = (p_i, q_i) \neq \infty$. When $d_i = \infty$ the torus T_i is cut out. Let M be an irreducible orientable closed 3-manifold and let $\mathcal{T}_M$ denote its family of canonical tori. From now on we adopt the following convention.

- For each T in $\partial\mathcal{S}(M)$ we fix a Seifert fibered space S_T adjacent to T and a basis (h_T, δ_T) of $\pi_1(T)$ where h_T corresponds to the generic fiber $h(S_T)$ of S_T and δ_T corresponds to the boundary component $\delta(S_T)$ of the 2-orbifold of S_T lying in $T \subset \partial S_T$. If S_T is adjacent to a Seifert fibered space S'_T along T we denote by $(h(S'_T), \delta(S'_T))$ an other basis for $\pi_1 T$ with respect to S'_T in the same way as for S_T. We denote by $d(S_T) = (a(S_T), b(S_T))$ the element of $\mathcal{P}_*$ such that $h(S'_T) = a(S_T) h_T + b(S_T)\delta_T$.

Note that $b(S_T) \neq 0$ by minimality of $\mathcal{T}_M$. Denote by $\mathcal{P}^0_*$ the finite subset of $\mathcal{P}_*$ defined by $\{(a(S_T), b(S_T)),\ T \in \partial\mathcal{S}(M) \setminus \partial\mathcal{S}(M) \cap \partial\mathcal{H}(M), (1,0), \infty\}$. Now to prove the Key Lemma we will show the following result.

Lemma 3.1. *Let $f : M \to N$ be a non-zero degree map between two closed Haken 3-manifolds such that $\mathrm{Vol}(M) = \deg(f)\,\mathrm{Vol}(N)$. Then for any component S in $\mathcal{S}(N)$ there exists a graph submanifold G_S (that can be chosen connected) in M such that if $T_1, \ldots, T_k$ denotes its boundary components then there exists $d_1, \ldots, d_k$ in $\mathcal{P}^0_*$ such that S is properly dominated by $(G_S)_{d_1,\ldots,d_k}$.*

Note that in Lemma 3.1 the surgery coefficients ∞ correspond always to π_1-injective tori in $\mathcal{T}_M$ for the map f. Lemma 3.1 implies the Key Lemma. Indeed it is sufficient to consider the set $\mathcal{G}_1$ which consists of all connected graph submanifolds of M whose boundaries are made of components of $\mathcal{T}_M$ and then performing all Dehn fillings among all elements of $\mathcal{G}_1$ with surgery coefficients equal to $(1,0), \infty$ or $(a(S_T), b(S_T))$ when $T \in \partial\mathcal{S}(M) \setminus \partial\mathcal{S}(M) \cap \partial\mathcal{H}(M)$ to get the finite set of 3-manifolds $\mathcal{G}$.

3.1. On the proof of Lemma 3.1

The first step is to prove that Lemma 3.1 is true for non-zero degree maps $f : M \to N$ such that $\mathrm{Vol}(M) = \deg(f)\,\mathrm{Vol}(N)$ and satisfying $f(\mathcal{S}(M)) \subset \mathrm{int}(\mathcal{S}(N))$. Hypothesis are the same as in Lemma 3.1.

Lemma 3.2. *If $f(\mathcal{S}(M)) \subset \mathrm{int}(\mathcal{S}(N))$ then for any component S in $\mathcal{S}(N)$ there exists a graph submanifold G_S (that can be chosen connected) in M which properly dominates S. In particular this means that in this case Lemma 3.1 is true with $\mathcal{P}^0_* = \{\infty\}$.*

We now give an outline of the proof of Lemma 3.1 in the general case. If S is a component of $\mathcal{S}(M)$ we denote by t_S the homotopy class of its regular fiber. Note that if for every S in $\mathcal{S}(M)$ the map $f|S:S\to N$ is non-degenerate in the sense of Jaco and Shalen (see [4]) then by [4, Mapping Theorem] we can invoque Lemma 3.2. If not, the main purpose of the proof is to realize a *good factorization* on the map f, which is inspired from a construction of Rong in [8, pp. 422–424], to have a reduction to the case of Lemma 3.2. More precisely let B_0 be the union of all S in $\mathcal{S}(M)$ such that $f|S$ is degenerate. If $f|S$ is a degenerate map then either $f_*(\pi_1 S)=\{1\}$ (case 1) or $f_*(\pi_1 S)=\mathbf{Z}$ (case 2) or, since $\pi_1(N)$ is torsion free, $(f|S)_*:\pi_1 S\to\pi_1 N$ factors through $\pi_1 V$, where V is the base 2-manifold of the Seifert fibered space S (case 3). Set $G_0=\overline{M-B_0}$. Define a subset of B_0 by setting $\mathcal{S}_0=\{S\in B_0$ s.t. S is adjacent to G_0 and $f_*(t_S)\neq 1\}$ and set $B_1=B_0-\mathcal{S}_0$ and $G_1=\overline{M-B_1}$. We continue this process by setting $\mathcal{S}_1=\{S\in B_1$ s.t. S is adjacent to G_1 and $f_*(t_S)\neq 1\}$ to construct a sequence $G_0\subset G_1\subset\cdots\subset G_i\subset G_{i+1}\subset\cdots$ which satisfies the following conditions:

(i) number of connected components n_{i+1} of $G_{i+1}\leqslant$ number of connected components n_i of G_i,
(ii) for any i, $\mathrm{int}(G_i)$ contains $\mathcal{H}(M)$,
(iii) for any i there exists a non-zero degree map $\beta_i:\widehat{G_i}\to N$, where $\widehat{G}_i$ denotes the space obtained from G_i after Dehn filling, such that $\deg(\beta_i)=\deg(f)$,
(iv) $\widehat{G}_i$ is still a Haken manifold and for each component S of $\mathcal{S}(\widehat{G}_i)$ the map $\beta_i|S:S\to N$ is homotopic to a map β_i' such that $\beta_i'(S)\subset\mathrm{int}(\mathcal{S}(N))$.

For these reasons G_i is called an essential part of M with respect to f. This process must stop. Set $n_0=\min\{n\geqslant 0$ such that $G_n=G_{n+1}\}$. Set G the maximal essential part G_{n_0} of M, $B=M-G$ and $\beta=\beta_{n_0}:\widehat{G}_{n_0}=\widehat{G}\to N$. Let $T_1,\ldots,T_l$ be the components of $\partial G=\partial B$ and let $S_1,\ldots,S_l$ (resp. $B_1,\ldots,B_l$) be the Seifert pieces (not necessarily pairwise distinct) in G (resp. in B) such that for each $i=1,\ldots,k$, B_i and S_i are adacent along T_i. We know from our construction that $f_*(h_{B_i})=1$, where h_{B_i} denotes the regular fiber of B_i. Let δ_{B_i} be the homotopy class of the boundary of the 2-orbifold of B_i lying in T_i. Denote by h_{S_i} the regular fiber of S_i and by $(a_{T_i}(S_i),b_{T_i}(S_i))$ the elements of $\mathcal{P}_*$ such that $h_{B_i}=a_{T_i}(S_i).h_{S_i}+b_{T_i}(S_i).\delta_{S_i}$ where δ_{S_i} is the homotopy class of the boundary of the 2-orbifold of S_i lying in T_i. If $(h_{T_i},\delta_{T_i})=(h_{B_i},\delta_{B_i})$ (i.e., if $S_{T_i}=B_i$) we set $d_i=(1,0)\in\mathcal{P}_*^0$ and if $(h_{T_i},\delta_{T_i})=(h_{S_i},\delta_{S_i})$ (i.e., $S_{T_i}=S_i$) then $(a_{T_i}(S_i),b_{T_i}(S_i))=(a(S_{T_i}),b(S_{T_i}))\in\mathcal{P}_*^0$ and we set $d_i=(a(S_{T_i}),b(S_{T_i}))\in\mathcal{P}_*^0$. Thus we get $\widehat{G}=G_{d_1,\ldots,d_l}$. Hence to complete the proof of Lemma 3.1 it remains to check that the map $\beta:\widehat{G}\to N$ satisfies the hypothesis of Lemma 3.2.

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