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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Convergence of the parabolic Ginzburg–Landau equation to motion by mean curvature

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Abstract

We present some new results for the asymptotic behavior of the complex parabolic Ginzburg–Landau equation. In particular, we establish that, as the parameter ε tends to 0, vorticity evolves according to motion by mean curvature in Brakke's weak formulation. The only assumption we make is a natural energy bound on the initial data. In some cases, we also prove convergence to enhanced motion in the sense of Ilmanen. ***To cite this article: F. Bethuel et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Convergence de l'équation de Ginzburg–Landau parabolique vers un mouvement par courbure moyenne. Nous présentons de nouveaux résultats concernant l'étude asymptotique du flot de la chaleur pour l'énergie de Ginzburg–Landau. En particulier, nous montrons que, lorsque le paramètre ε tend vers 0, la vorticité évolue selon un mouvement par courbure moyenne, dans un sens faible introduit par Brakke. Notre seule hypothèse concerne une borne naturelle portant sur l'énergie de la condition initiale. Dans certains cas, nous montrons également la convergence vers un mouvement par courbure moyenne dans un sens plus fort dû à Ilmanen. ***Pour citer cet article : F. Bethuel et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Nous présentons dans cette Note les principaux résultats obtenus dans [7] pour l'étude asymptotique, lorsque le paramètre ε tend vers zéro, du flot de la chaleur $(\mathrm{PGL})_\varepsilon$ associé à l'énergie de Ginzburg–Landau $\mathcal{E}_\varepsilon$ (définie par la formule (1)). Notre seule hypothèse concerne l'énergie de la donnée initiale u_ε^0 : nous supposons qu'elle vérifie une borne du type $\mathcal{E}_\varepsilon(u_\varepsilon^0) \leqslant M_0|\log\varepsilon|$. Une telle borne est naturelle dans le contexte ; en effet, en dimension deux, l'énergie d'un tourbillon de vorticité est de l'ordre de $|\log\varepsilon|$ (voir, e.g., [3]).

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Nous avons porté principalement notre attention sur le comportement de la concentration locale d'énergie, décrite par la fonction $e_\varepsilon(u_\varepsilon)$ (définie dans (1)), et les mesures « renormalisées » associées $\mu_\varepsilon^t(x)$, définies dans (4) et (5). Au vu des inégalités d'énergie classiques (2), (3), ces mesures sont bornées (grâce à la renormalisation par le facteur $|\log\varepsilon|^{-1}$). Elles convergent donc au sens des mesures (pour des sous-suites) vers des mesures limite μ_*^t.

Notre premier résultat concerne la structure de la mesure limite μ_*^t, décrite de manière abstraite dans la formule (6) : nous montrons qu'elle se décompose en deux termes distincts. Le premier désigne la partie absolument continue par rapport à la mesure de Lebesgue. Sa densité est donnée par la fonction régulière $|\nabla\Phi_*|^2(\cdot,t)$, où la fonction Φ_* est une solution de l'équation de la chaleur homogène sur $\mathbb{R}^N\times\mathbb{R}^+$. Nous appelons ce mode de l'énergie le mode linéaire. Le second terme correspond à la partie « concentrée » de l'énergie, localisée sur un ensemble rectifiable Σ_μ^t de dimension $N-2$. Celui-ci représente également l'ensemble de concentration de la vorticité. Comme cette dernière est essentiellement de nature topologique, nous appelons ce second mode de l'énergie le mode topologique.

Notre second résultat concerne l'évolution du mode topologique (le mode linéaire, comme nous l'avons déjà dit, est régi par l'équation de la chaleur). Nous montrons que la partie concentrée de la mesure ν_*^t (définie dans (7)) évolue par courbure moyenne, au sens faible introduit par Brakke [10].

En résumant, nos résultats montrent essentiellement que l'énergie se propage au cours du temps par deux modes de natures très différentes. Le point essentiel est que chacun des modes obéit à une équation propre, et qu'ils n'interagissent donc pas dans le régime d'énergie considéré ici.

In this Note, we consider the asymptotic analysis, as the parameter ε goes to zero, of the complex-valued parabolic Ginzburg–Landau equation for functions $u_\varepsilon:\mathbb{R}^N\times\mathbb{R}^+\to\mathbb{C}$ in space dimension $N\geqslant 3$, namely

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}-\Delta u_\varepsilon=\dfrac{1}{\varepsilon^2}u_\varepsilon\big(1-|u_\varepsilon|^2\big) & \text{on } \mathbb{R}^N\times(0,+\infty),\\ u_\varepsilon(x,0)=u_\varepsilon^0(x) & \text{for } x\in\mathbb{R}^N. \end{cases} \tag{PGL$_\varepsilon$}$$

It corresponds to the heat-flow for the Ginzburg–Landau energy

$$\mathcal{E}_\varepsilon(u)=\int_{\mathbb{R}^N} e_\varepsilon(u)=\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{2}+V_\varepsilon(u)\quad \text{for } u:\mathbb{R}^N\to\mathbb{C}, \tag{1}$$

where V_ε denotes the nonconvex potential

$$V_\varepsilon(u)=\frac{(1-|u|^2)^2}{4\varepsilon^2}.$$

This energy plays an important role in physics, and has been studied extensively from the mathematical point of view in the last decades. It is well known that $(\mathrm{PGL})_\varepsilon$ is well-posed for initial datas in H^1_{loc} with finite Ginzburg–Landau energy $\mathcal{E}_\varepsilon(u_\varepsilon^0)$. Moreover, we have the energy identity

$$\mathcal{E}_\varepsilon\big(u_\varepsilon(\cdot,T_2)\big)+\int_{T_1}^{T_2}\int_{\mathbb{R}^N}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t=\mathcal{E}_\varepsilon\big(u_\varepsilon(\cdot,T_1)\big)\quad \forall 0\leqslant T_1\leqslant T_2. \tag{2}$$

We assume that the initial condition u_ε^0 verifies the bound, natural in this context,

$$\mathcal{E}_\varepsilon\big(u_\varepsilon^0\big)\leqslant M_0|\log\varepsilon|, \tag{H$_0$}$$

where M_0 is a fixed positive constant. Therefore, in view of (2) we have

$$\mathcal{E}_\varepsilon\big(u_\varepsilon(\cdot,T)\big)\leqslant\mathcal{E}_\varepsilon\big(u_\varepsilon^0\big)\leqslant M_0|\log\varepsilon|\quad \text{for all } T\geqslant 0. \tag{3}$$

The main emphasis of this note is placed on the asymptotic limits of the Radon measures μ_ε defined on $\mathbb{R}^N \times [0, +\infty)$ by

$$\mu_\varepsilon(x,t) = \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log\varepsilon|}\,\mathrm{d}x\,\mathrm{d}t, \tag{4}$$

and of their time slices μ_ε^t defined on $\mathbb{R}^N \times \{t\}$ by

$$\mu_\varepsilon^t(x) = \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log\varepsilon|}\,\mathrm{d}x, \tag{5}$$

so that $\mu_\varepsilon = \mu_\varepsilon^t\,\mathrm{d}t$. In view of assumption (H_0) and (3), we may assume, up to a subsequence $\varepsilon_n \to 0$, that there exists a Radon measure μ_* defined on $\mathbb{R}^N \times [0, +\infty)$ such that

$$\mu_\varepsilon \rightharpoonup \mu_* \quad \text{as measures.}$$

Actually, passing possibly to a further subsequence, we may also assume, using a classical semi-decreasing property, that

$$\mu_\varepsilon^t \rightharpoonup \mu_*^t \quad \text{as measures on } \mathbb{R}^N \times \{t\},\ \text{for all } t \geqslant 0.$$

Our main results describe the properties of the measures μ_*^t. Complete proofs are presented in [7]. We first have

Theorem 1. *There exist a subset Σ_μ in $\mathbb{R}^N \times (0, +\infty)$, and a smooth real-valued function Φ_* defined on $\mathbb{R}^N \times (0, +\infty)$ such that the following properties hold.*

(i) *Σ_μ is closed in $\mathbb{R}^N \times (0, +\infty)$ and for any compact subset $\mathcal{K} \subset \mathbb{R}^N \times (0, +\infty) \setminus \Sigma_\mu$*

$$\left|u_\varepsilon(x,t)\right| \to 1 \quad \textit{uniformly on } \mathcal{K} \textit{ as } \varepsilon \to 0.$$

(ii) *For any $t > 0$, $\Sigma_\mu^t \equiv \Sigma_\mu \cap \mathbb{R}^N \times \{t\}$ verifies*

$$\mathcal{H}^{N-2}\big(\Sigma_\mu^t\big) \leqslant K M_0.$$

(iii) *The function Φ_* verifies the heat equation on $\mathbb{R}^N \times (0, +\infty)$.*

(iv) *For each $t > 0$, the measure μ_*^t can be exactly decomposed as*

$$\mu_*^t = |\nabla\Phi_*|^2\mathcal{H}^N + \Theta_*(x,t)\mathcal{H}^{N-2} \llcorner \Sigma_\mu^t, \tag{6}$$

where $\Theta_*(\cdot, t)$ is a bounded function.

(v) *There exists a positive function η defined on $\mathbb{R}_*^+$ such that, for almost every $t > 0$, the set Σ_μ^t is $(N-2)$-rectifiable and*

$$\Theta_*(x,t) = \Theta_{N-2}\big(\mu_*^t, x\big) = \lim_{r\to 0}\frac{\mu_*^t(B(x,r))}{\omega_{N-2}r^{N-2}} \geqslant \eta(t),$$

for $\mathcal{H}^{N-2}$ a.e. $x \in \Sigma_\mu^t$.

In view of the decomposition (6), μ_*^t can be split into two parts. A diffuse part $|\nabla\Phi_*|^2$, and a concentrated part

$$\nu_*^t = \Theta_*(x,t)\mathcal{H}^{N-2} \llcorner \Sigma_\mu^t. \tag{7}$$

By (iii), the diffuse part is governed by the heat equation. We therefore call it the "linear mode". On the other hand, ν_*^t essentially corresponds to the concentration of vorticity, and we call it the topological mode. Our next theorem focuses on the evolution of the topological mode as time varies.

Theorem 2. *The family $(\nu_*^t)_{t>0}$ is a mean curvature flow in the sense of Brakke* [10].

Comment. We recall that there exists a classical notion of mean curvature flow for smooth compact embedded manifolds. In this case, the motion corresponds basically to the gradient flow for the area functional. It is well known that such a flow exists for small times (and is unique), but develops singularities in finite time. Brakke [10] introduced a weak formulation which allows to encompass singularities and makes sense for (rectifiable) measures. Whereas it allows to handle a large class of objects, an important and essential flaw of Brakke's formulation is that there is never uniqueness. Even though nonuniqueness is presumably an intrinsic property of mean curvature flow, a major part of nonuniqueness in Brakke's formulation is nonintrinsic, and therefore allows for weird solutions. A stronger notion of solution will be discussed in Theorem 4.

The proof of Theorem 2 relies both on the measure theoretic analysis of Ambrosio and Soner [1], and on the analysis of the structure of μ_*, in particular the statements in Theorem 1. In [1], Ambrosio and Soner proved the result in Theorem 2 under the additional assumption

$$\limsup_{r\to 0} \frac{\mu_*^t(B(x,r))}{\omega_{N-2} r^{N-2}} \geqslant \eta, \quad \text{for } \mu_*^t\text{-a.e. } x, \tag{8}$$

for some constant $\eta > 0$. In view of the decomposition (6), assumption (8) holds if and only if $|\nabla \Phi_*|^2$ vanishes, i.e., there is no diffuse energy. If $|\nabla \Phi_*|^2$ vanishes, it follows therefore that Theorem 2 can be directly deduced from [1], Theorem 5.1, and statements (iv) and (v) in Theorem 1.

In the general case where $|\nabla \Phi_*|^2$ does not vanish, their argument has to be adapted, however without major changes. Indeed, one of the important consequences of our analysis in [7] is that the concentrated and diffuse energies do not interfere, i.e., the two modes for the energy are completely decoupled.

In view of the previous discussion, one may wonder if some conditions on the initial data will guarantee that there is no diffuse part. In this direction, we introduce the conditions

$$u_\varepsilon^0 \equiv 1 \quad \text{in } \mathbb{R}^N \setminus B(R_1) \tag{H$_1$}$$

for some $R_1 > 0$, and

$$\left\| u_\varepsilon^0 \right\|_{H^{1/2}(B(R_1))} \leqslant M_2. \tag{H$_2$}$$

Theorem 3. *Assume that u_ε^0 verifies* (H_0), (H_1) *and* (H_2). *Then $|\nabla \Phi_*|^2 = 0$, and the family $(\mu_*^t)_{t>0}$ is a mean curvature flow in the sense of Brakke.*

In stating conditions (H_1) and (H_2) we have not tried to be exhaustive, and there are many ways to generalize them.

We now come back to the already mentioned difficulty related to Brakke's weak formulation, namely the strong nonuniqueness. To overcome this difficulty, Ilmanen [16] introduced the stronger notion of enhanced motion, which applies to a slightly smaller class of objects, but has much better uniqueness properties (see [16]). In this direction we prove the following.

Theorem 4. *Let $\mathcal{M}_0$ be any given integer multiplicity $(N-2)$-current without boundary, with bounded support and finite mass. There exists a sequence $(u_\varepsilon^0)_{\varepsilon>0}$ and an integer multiplicity $(N-1)$-current $\mathcal{M}$ in $\mathbb{R}^N \times [0, +\infty)$ such that*

$$\text{(i)} \quad \partial\mathcal{M} = \mathcal{M}_0, \qquad \text{(ii)} \quad \mu_*^0 = \pi|\mathcal{M}_0|,$$

and such that the pair $(\mathcal{M}, \frac{1}{\pi}\mu_*^t)$ is an enhanced motion in the sense of Ilmanen [16].

Remark 1. Our result is a little stronger than the statement of Theorem D. Indeed, we show that *any* sequence u_ε^0 satisfying $Ju_\varepsilon^0 \rightharpoonup \pi\mathcal{M}_0$ and $\mu_*^0 = \pi|\mathcal{M}_0|$ gives rise to an Ilmanen motion. (Ju_ε^0 denotes the Jacobian of u_ε^0.)

Eq. $(\mathrm{PGL})_\varepsilon$ has already been considered in recent years. In particular, the dynamics of vortices has been described in the two dimensional case (see [17,20]). Concerning higher dimensions $N \geqslant 3$, under the assumption that the initial measure is concentrated on a smooth manifold, a conclusion similar to ours has been obtained first on a formal level by Pismen and Rubinstein [25], and then rigorously by Jerrard and Soner [18] and Lin [21], only in the time interval where the classical solution exists, that is before the appearance of singularities. As already mentioned, a first convergence result past the singularities has been obtained by Ambrosio and Soner [1], under the crucial density assumption (8) for the measures μ_*^t discussed above. Some important asymptotic properties for solutions of $(\mathrm{PGL})_\varepsilon$ were also considered in [24,29,5].

Beside these works, we had at least two important sources of inspiration in our study. The first one was the corresponding theory for the elliptic case, developed in the last decade, in particular in [3,28,8,26,22,23,4,19,9,6]. The second one was the corresponding theory for the scalar case (i.e., the Allen–Cahn equation) developed in particular in [11,13,12,14,15,27]. As in this case, the Clearing-Out Lemma is a step-stone in the proofs of Theorems 1–4.

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