Available online at www.sciencedirect.com
SCIENCE @ DIRECT®
C. R. Acad. Sci. Paris, Ser. I 336 (2003) 419–422
COMPTES RENDUS MATHEMATIQUE

Differential Geometry

Geometric Anosov flows of dimension 5

Flots d'Anosov géométriques de dimension 5

Yong Fang

Laboratoire de mathématique d'Orsay, UMR 8628 du CNRS, Université Paris-Sud, France

Received 20 January 2003; accepted 30 January 2003

Presented by Jean-Michel Bismut

Abstract

We show that for a smooth Anosov flow on a closed five dimensional manifold, if it has C^∞ Anosov splitting and preserves a C^∞ pseudo-Riemannian metric, then up to a special time change and finite covers, it is C^∞ flow equivalent either to the suspension of a symplectic hyperbolic automorphism of $\mathbb{T}^4$, or to the geodesic flow on a three dimensional hyperbolic manifold. ***To cite this article: Y. Fang, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous classifions les flots d'Anosov lisses sur des variétés fermées de dimension 5, qui préservent une métrique pseudo-Riemannienne lisse et dont les distributions d'Anosov sont C^∞. A un changement du temps spécial et un revêtement fini près, un tel flot est C^∞ conjugué ou bien, à une suspension d'un automorphisme hyperbolique symplectique de $\mathbb{T}^4$, ou bien à un flot géodésique sur une variété hyperbolique de dimension 3. ***Pour citer cet article : Y. Fang, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

Let M be a C^∞-closed manifold with a Riemannian metric. A C^∞ flow, ϕ_t, generated by a non-singular vector field X is called an Anosov flow, if there exists a ϕ_t-invariant splitting $TM = \mathbb{R}X \oplus E^+ \oplus E^-$ and two positive numbers a and b, such that

$$\forall u^\pm \in E^\pm,\ \forall t \geqslant 0, \quad \left\| D\phi_{\mp t}\left(u^\pm\right)\right\| \leqslant a\,\mathrm{e}^{-bt}\left\|u^\pm\right\|.$$

In general, the subbundles E^+ and E^- are only continuous and rarely smooth (see [4] and [2]). If E^+ and E^- are smooth and ϕ_t preserves in addition a C^∞ pseudo-Riemannian metric, then the flow is called *geometric*.

Let ϕ_t be a *geometric* Anosov flow, preserving a pseudo-Riemannian metric g. The flow preserves a C^∞ 1-form λ, such that $\lambda(X) = 1$ and $\lambda(E^\pm) = 0$. Let J be the section of $(TM)^* \otimes TM$, such that $J(X) = 0$ and $J(u^\pm) = \pm u^\pm$. Then $\omega := g(J\cdot,\cdot)$ is a ϕ_t-invariant 2-form with $\mathbb{R}X$ as kernel. Since $E^\pm$ are both Lagrangian for

E-mail address: fangyong1@yahoo.fr (Y. Fang).

1631-073X/03/$ – see front matter © 2003 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS. Tous droits réservés.
doi:10.1016/S1631-073X(03)00096-7

$\omega|_{E^+\oplus E^-}$, then $\dim E^+ = \dim E^- =: n$. The volume form $\lambda \wedge (\wedge^n \omega)$ is preserved by ϕ_t. So the flow is topologically transitive.

Definition. $\mathrm{rank}(\phi_t) := 2(\max\{k \geqslant 0 \mid \wedge^k \mathrm{d}\lambda \not\equiv 0\})$.

This even number is called the *rank* of the flow. We have obviously $0 \leqslant \mathrm{rank}(\phi_t) \leqslant 2n$. By combining the results of [3] and [6], we easily get the following:

Theorem 1. *Let M be a C^∞-closed manifold of dimension $2n+1$, and ϕ_t be a geometric Anosov flow on M, we have*

(i) *if $\mathrm{rank}(\phi_t) = 0$, then, up to a constant change of time scale, ϕ_t is C^∞ flow equivalent to the suspension of a hyperbolic infranilautomorphism*;
(ii) *if $\mathrm{rank}(\phi_t) = 2n$, then, up to finite covers, ϕ_t is C^∞ flow equivalent to a canonical perturbation of the geodesic flow on a locally symmetric Riemannian manifold of strictly negative curvature.*

A canonical perturbation of a geodesic flow with generator X is (by definition) the flow of the field $X/(1+\alpha(X))$, where α is a C^∞ closed 1-form such that $1+\alpha(X) > 0$.

In the case of dimension 5, we prove the following

Theorem 2. *Let M be a closed manifold of dimension 5, and ϕ_t be a geometric Anosov flow on M, then*

(i) *either, up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent to the suspension of a symplectic hyperbolic automorphism of $\mathbb{T}^4$*;
(ii) *or, up to finite covers, ϕ_t is C^∞ flow equivalent to a canonical perturbation of the geodesic flow on a three-dimensional Riemannian manifold of constant negative curvature.*

2. Proof of Theorem 2

By Theorem 1, we need only eliminate the case of rank 2, i.e., $\mathrm{d}\lambda \not\equiv 0$ and $\mathrm{d}\lambda \wedge \mathrm{d}\lambda \equiv 0$. Suppose on the contrary the existence of such a flow ϕ_t. In this section, this flow will be proved to be homogeneous. Then in the following sections, we shall eliminate the possible homogeneous models by some dynamical and Lie theoretical arguments.

Define $U := \{x \in M \mid \mathrm{d}\lambda(x) \neq 0\}$, $E_1 := \{y \in E^+ \oplus E^- \mid i_y \,\mathrm{d}\lambda = 0\}$. Since $\mathrm{d}\lambda \not\equiv 0$ and $\mathrm{d}\lambda \wedge \mathrm{d}\lambda \equiv 0$, then E_1 is a 2-dimensional C^∞ subbundle of $TM|_U$. Denote the probability of the volume form $\lambda \wedge \omega \wedge \omega$ by ν.

Lemma 2.1. *The Lyapunov decomposition of ϕ_t, with respect to ν, is smooth.*

Proof. If b is a Lyapunov exponent of ϕ_t, then so is $-b$. If ϕ_t has two positive Lyapunov exponents, $E_1 \cap E^\pm$ coincide with two of the Lyapunov subbundles on a conull subset. To get the others, we take their dual with respect to ω. $\square$

Now we can introduce a C^∞ connection ∇ on M, adapted to the Lyapunov decomposition of ϕ_t, such that

$$\nabla X = 0, \qquad \nabla \omega = 0, \qquad \nabla E_i^\pm \subseteq E_i^\pm,$$
$$\nabla_{Z_j^\pm} Z_i^\mp = P_i^\mp\big[Z_j^\pm, Z_i^\mp\big], \qquad \nabla_X Z_i^\pm = \big[X, Z_i^\pm\big] \pm \alpha_i Z_i^\pm,$$

where $E_i^\pm$ are Lyapunov subbundles with Lyapunov exponents $\pm\alpha_i$, and $P_i^\pm$ are the projections of TM onto $E_i^\pm$.

Lemma 2.2. *Let $\widetilde{M}$ be the universal cover of M, then the group of diffeomorphisms of $\widetilde{M}$, which preserve $\widetilde{X}$, $\widetilde{\omega}$ and the lifted Lyapunov decomposition, is a Lie group and acts transitively on $\widetilde{M}$.*

Proof. By the definition of ∇, $\nabla R = 0$, $\nabla T = 0$ and ∇ is complete (see [5]). $\square$

Denote the previous Lie group by G. Fix a point $x \in \widetilde{M}$ and denote the isotropy group of x by H. Then G/H is a reductive homogeneous space with ∇ as its canonical connection. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, then $\mathfrak{g} \cong \mathfrak{h} \oplus T_x\widetilde{M}$. By the linear isotropy representation, $\mathfrak{h}$ is seen to be isomorphic to $\mathbb{R}$ or $\mathbb{R}^2$. We deduce that G is simply connected. Denote the fundamental group by Γ, then up to finite covers, we can suppose that $\Gamma \subseteq G_e$, where G_e denotes the connected component of the unit of G. ∇ induces a connection ∇^+ on $\wedge^2 E^+$. Denote the connection form and the curvature form of ∇^+ by β^+ and Ω^+.

Lemma 2.3. $\mathrm{d}\lambda \wedge \Omega^+ = 0$, $\Omega^+ \wedge \Omega^+ = 0$, $\Omega^+ \wedge \omega = 0$.

3. Suppose at first that ϕ_t has two positive Lyapunov exponents. Then by the previous lemma, $\Omega^+ = 0$.

3.1. If $\dim(\mathfrak{h}) = 1$, then by a direct calculation and [1], up to finite covers, $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}$ and Γ is identified to a cocompact lattice of this group. Here the semi-direct product is given by the linear action. But we can easily see that $\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}$ admits no cocompact lattice. So this case is impossible.

3.2. If $\dim(\mathfrak{h}) = 2$, then by a direct calculation, $G_e \cong (\mathbb{R}^2 \rtimes \mathbb{R}) \times \widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathbb{R}$. The space of weak unstable leaves is seen to be $\mathbb{R}^2$. Using the density of periodic orbits of ϕ_t in M, we can find an element $\gamma \in \Gamma \cap (\mathrm{Cent}(G_e))^c$, which acts with a saddle on the space of weak unstable leaves. But it is impossible (see [2], 5).

4. Suppose that ϕ_t has only one positive Lyapunov exponent and $\mathrm{d}\lambda \wedge \omega \not\equiv 0$.

4.1. If $\dim(\mathfrak{h}) = 2$, then by the linear isotropy representation of H, $\Omega^+ = 0$. In this case, we get the same groups G_e and H_e as in 3.2. So the same arguments prove the non-existence of this case.

4.2. If $\dim(\mathfrak{h}) = 1$, then $\widetilde{M}$ can be identified to $\mathbb{R}^2 \times \mathrm{Heis}$, where Heis is the 3-dimensional Heisenberg group. In this case, we can find a group of automorphisms of $\mathbb{R}^2 \times \mathrm{Heis}$, which is isomorphic to $\mathbb{R}^2$ and acts on $\widetilde{M}$ as the isometries of the geometric structure $(\widetilde{X}, \widetilde{E}^{\pm}, \widetilde{\omega})$. We deduce that $\dim(\mathfrak{h}) \geqslant 2$, which is a contradiction.

5. Suppose that ϕ_t has one positive Lyapunov exponent and $\mathrm{d}\lambda \wedge \omega \equiv 0$. If $\dim(\mathfrak{h}) = 1$, then the same argument as in 4.2. gives a contradiction. So we suppose that $\dim(\mathfrak{h}) = 2$. By Lemma 2.3, $\exists c \in \mathbb{R}$, such that $\Omega^+ = c \cdot \mathrm{d}\lambda$.

Let J be the section of $(TM)^* \otimes TM$ defined in the introduction. Construct a new connection $\nabla_1 := \nabla - \frac{c}{2}\lambda \otimes J$, then $\Omega_1^+ = 0$, where Ω_1^+ is the curvature form of the induced connection of ∇_1 on $\wedge^2 E^+$. We have also $\nabla_1 R^{\nabla_1} = 0$, $\nabla_1 T^{\nabla_1} = 0$, and $\nabla_1 \omega = 0$. Let G be the isometry group of $(\widetilde{X}, \widetilde{E}^{\pm}, \widetilde{\omega})$ and H be the isotropy subgroup of x as above. Then using the horizontal distribution of ∇_1, we get another identification $\mathfrak{g} \cong T_x\widetilde{M} \oplus \mathfrak{h}$.

Now by a direct calculation and the Anosov property, $\widetilde{M}$ is identified to $(\mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}_0(1,2)})/\mathbb{R}$, where $\mathrm{SO}_0(1,2)$ is the connected component of Id of the isometry group of the quadratic form: $-\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. The semi-direct product is given by the natural linear action and $\mathbb{R}$ is the 1-parameter subgroup generated by $((0,0,1),0) \in$

$\mathbb{R}^3 \rtimes \mathfrak{so}(1,2)$. Γ is identified with a discrete subgroup of $\mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}_0(1,2)}$. We finish the proof of Theorem 2 by proving

Lemma 5.1. $\mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}_0(1,2)}$ *admits no discrete subgroup, which acts properly, freely, and cocompactly on* $(\mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}_0(1,2)})/\mathbb{R}$.

Proof. Suppose the existence of such a discrete subgroup, denoted by Γ_1. Then Γ_1 is seen to be non-solvable. Since the action of $\widetilde{\mathrm{SO}_0(1,2)}$ on $\mathbb{R}^3$ is irreducible, then Γ_1 is Zariski-dense in $\mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}_0(1,2)}$. Let Δ be the projection of Γ_1 into $\widetilde{\mathrm{SO}_0(1,2)}$. Then by [7], Δ is discrete in $\widetilde{\mathrm{SO}_0(1,2)}$. We deduce that $\mathbb{R}^3$ acts properly on $\mathbb{R}^3/\mathbb{R}$, which is absurd. $\square$

Acknowledgements

The author would like to thank P. Foulon, P. Pansu, Y. Benoist, and F. Labourie for discussions and help.

References

[1] Y. Benoist, P. Foulon, F. Labourie, Flots d'Anosov à distributions de Liapounov différentiables, I, Ann. Inst. H. Poincaré 53 (1990) 395–412.
[2] Y. Benoist, P. Foulon, F. Labourie, Flots d'Anosov à distributions stable et instable différentiables, J. Amer. Math. Soc. 5 (1992) 33–74.
[3] Y. Benoist, F. Labourie, Sur les difféomorphismes d'Anosov affines à feuilletages stable et instable différentiables, Invent. Math. 111 (1993) 285–308.
[4] R. Feres, A. Katok, Anosov flows with smooth foliations and rigidity of geodesic flows on three-dimensional manifolds of negative curvature, Ergodic Theory Dynamical Systems 10 (1990) 657–670.
[5] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vol. I, II, Interscience, New York, 1963.
[6] J.F. Plante, Anosov flows, Amer. J. Math. 94 (1972) 729–754.
[7] M.S. Raghunathan, Discrete Subgroups of Lie Groups, Springer, Berlin, 1972.