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A reduction of the slicing problem to finite volume ratio bodies

Réduction du problème des sections de corps convexes au cas de rapport volumique borné

Jean Bourgain [a], Bo'az Klartag [b,1], Vitali Milman [b]

[a] *Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA*
[b] *School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel*

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Abstract

Here we discuss results around the slicing problem, which is a well known open problem in asymptotic convex geometry. We show that if one can prove that the isotropic constant of bodies with a finite volume ratio is uniformly bounded – then it would follow that the isotropic constant of any convex body is uniformly bounded. ***To cite this article: J. Bourgain et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Cette Note concerne le problème bien connu de la minoration uniforme de la mesure des sections de codimension 1 de corps convexes isotrope dans $\mathbb{R}^n$, ce qui équivaut à une borne uniforme de la constante d'isotropie. Nous démontrons qu'une réponse affirmative à cette question dans le cas particulier d'un corps à rapport volumique borné (c'est-à-dire tel que la racine n-ième du volume de l'ellipsoide de John admet une borne inférieure) entraîne une réponse affirmative en général. La méthode utilise des techniques de symétrisation et de géométrie des espaces de Banach. ***Pour citer cet article : J. Bourgain et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

Let $K \subset \mathbb{R}^n$ be a convex body of volume one whose barycenter is at the origin (i.e., $b(K) = \int_K \vec{x}\,\mathrm{d}x = 0$). It is well known (see [6]) that there exists a unique positive definite linear transformation T with $\det(T) = 1$, such that for any unit vector $\theta \in S^{n-1}$,

E-mail addresses: bourgain@math.ias.edu (J. Bourgain), klartagb@post.tau.ac.il (B. Klartag), milman@post.tau.ac.il (V. Milman).
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$$\int_{T(K)} \langle x,\theta\rangle^2 \,\mathrm{d}x = L_K^2$$

independently of θ. The number L_K is referred to as the isotropic constant of the body K. If the transformation T is the identity map, we say that K is isotropic, or that it is in isotropic position.

It is a major unsolved problem, whether there exists a numerical constant C such that $L_K < C$ for every convex body in any finite dimension. A positive answer to this question has many interesting consequences, see [6]. Just to mention one, it implies that every convex body of volume one, has an $(n-1)$-dimensional section whose $(n-1)$-dimensional volume is greater than some constant c. The best estimate known today is $L_K < cn^{1/4}\log n$, for an arbitrary convex $K \subset \mathbb{R}^n$ (see [2], or the presentation in [3]). For certain classes of convex bodies the question is affirmatively answered, such as unconditional bodies (as observed by Bourgain, see [6]), zonoids, duals to zonoids (see [1]) or duals to bodies with a finite volume ratio (see [6]). Here we show a reduction of the general problem, to the boundedness of the isotropic constant of bodies with a finite volume ratio. For $K \subset \mathbb{R}^n$, the volume ratio of K is defined as,

$$v.r.(K) = \sup_{\mathcal{E}\subset K} \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(\mathcal{E})}\right)^{1/n},$$

where the supremum is over all ellipsoids contained in K. Formally, we prove the following conditional proposition:

Proposition 1.1. *There exists a number $v > 0$, such that if there exists a number c_1, such that for all n, and for all $K \subset \mathbb{R}^n$, inequality $v.r.(K) < v$ implies $L_K < c_1$, then there exists a numerical constant c_2 such that for all n, and for all $K \subset \mathbb{R}^n$ we have $L_K < c_2$. Furthermore, the dependence of c_2 on c_1 is almost linear. For any $\delta > 0$, there exist numbers $v(\delta), c(\delta) > 0$ such that if for any body $K \subset \mathbb{R}^n$ with $v.r.(K) < v(\delta)$, we have $L_K < u(n)$ – then for an arbitrary convex body $K \subset \mathbb{R}^n$, $L_K < c(\delta)u(n)^{1+\delta}$.*

2. Ideas used in the proof

The proof of Proposition 1.1 uses two powerful tools. The first is the classical method of symmetrization due to Steiner. We shall elaborate on this later on. The second important fact, is the existence of an M-ellipsoid (see [4], or Chapter 7 in the book [7]) in the following formulation:

Proposition 2.1. *Let $K \subset \mathbb{R}^n$ be any convex body. Then there exists an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ with $\mathrm{Vol}(\mathcal{E}) = \mathrm{Vol}(K)$ such that*

$$N(K,\mathcal{E}) = \min\{\sharp A;\, K \subset A + \mathcal{E}\} < \mathrm{e}^{cn},$$

where $\sharp A$ is the number of elements in the set A, and c is a numerical constant.

One of the consequences of the existence of an M-ellipsoid, is the fact that any convex body has at least one projection to a proportional dimension, which has a finite volume ratio. This result, formulated in the following lemma, appears originally in [5]. It can be also deduced from the proof of Corollary 7.9 in [7].

Lemma 2.2. *Let $K \subset \mathbb{R}^n$ be any convex body. Let $0 < \lambda < 1$. Then there exists a subspace G of dimension $\lfloor \lambda n \rfloor$ such that if $P : \mathbb{R}^n \to \mathbb{R}^n$ is a projection (i.e., P is linear and $P^2 = P$) with $G = \ker(P)$, then $P(K)$ has a volume ratio smaller than $c(\lambda)$, where $c(\lambda)$ is some function which depends solely on λ.*

We turn to describe Steiner symmetrization, in a generalization referred to as Schwartz–Steiner symmetrization. Let $K \subset \mathbb{R}^n$ be a convex body, and let $F \subset \mathbb{R}^n$ be any subspace of any dimension. Define the Schwartz–Steiner symmetrization of K with respect to F as the unique body K' such that for all $x \in F$ the body $K' \cap (x + F^{\perp})$ is a Euclidean ball centered at a point belonging to F and

$$\mathrm{Vol}\left(K \cap \left(x + F^{\perp}\right)\right) = \mathrm{Vol}\left(K' \cap \left(x + F^{\perp}\right)\right),$$

where Vol, of course, is the Euclidean volume in the corresponding subspace. It is easy to verify that K' is convex ("Brunn principle"), and that $\mathrm{Vol}(K') = \mathrm{Vol}(K)$. The role of the next lemma is to connect the isotropic constants of K and K'. Whenever we write $A \approx B$, we mean that there exist numerical constants $c_1, c_2 > 0$ such that $c_1 A < B < c_2 A$.

Lemma 2.3. *Let $K \subset \mathbb{R}^n$ be a convex isotropic body of volume one, and let K' be its Schwartz–Steiner symmetrization with respect to a k-codimensional subspace F. Then*

$$L_{K'} \approx \frac{L_K^{1-k/n}}{\mathrm{Vol}(\mathrm{Proj}_F(K))^{1/n}},$$

where Proj_F *is the orthogonal projection onto F in $\mathbb{R}^n$.*

We would like to combine the properties of an M-ellipsoid, together with the properties of the isotropic position (Lemma 2.3). This cannot be done in a direct manner, since apriori the M-ellipsoid and the isotropy ellipsoid may be very different. Our method to bypass this obstacle, is to show that for bodies with largest possible isotropic constant, these two ellipsoids coincide in some subspace of proportional dimension.

3. Worst possible body

Define $L_n = \sup_{C \subset \mathbb{R}^n} L_C$ where the supremum is over all convex sets of volume one in $\mathbb{R}^n$. Define K to be one of the worst possible bodies of dimension up to n, i.e., $L_K = \sup_{m \leqslant n} L_m$ and K is isotropic and of volume one. The dimension of K may be smaller than n (yet it isn't smaller than $n/2$). With some abuse of notation, we use the letter n to denote the dimension of K. One can apply the connection between the isotropic constant and volumes of sections of arbitrary dimension (that appears in [6], Proposition 3.11), to prove the following:

Lemma 3.1. *Let K be a worst possible isotropic body of volume one. Then for any subspace E of any dimension $1 \leqslant k \leqslant n$,*

$$\mathrm{Vol}\big(\mathrm{Proj}_E(K)\big)^{1/k} \geqslant c,$$

where c is some numerical constant.

Lemma 3.1 has some consequences about the structure of an M ellipsoid. Recall Proposition 2.1. K is covered by an exponential number of translations of $\mathcal{E}$. If we take $k = \lfloor \frac{n}{2} \rfloor$, and project K to a subspace E of dimension k, then $\mathrm{Vol}(\mathrm{Proj}_E(K)) \leqslant \mathrm{e}^{cn}\, \mathrm{Vol}(\mathrm{Proj}_E(\mathcal{E}))$ and we may conclude that projections of the M ellipsoid to any subspace of proportional dimension have large volume. Since $\mathcal{E}$ is an ellipsoid of volume one, this fact implies stringent conditions on the lengths of the axes of $\mathcal{E}$, that lead to the following:

Claim 3.2. *There exists a subspace E of dimension $\lceil \frac{n}{2} \rceil$ such that if we denote by D_E the Euclidean ball of volume one in E (centered at the origin), then*

$$\mathrm{Proj}_E(\mathcal{E}) \subset c D_E,$$

where c is some numerical constant.

4. Reducing to finite volume ratio

Let us look at the projection of the worst possible body K to the subspace E from Claim 3.2. According to Lemma 2.2, this body $\mathrm{Proj}_E(K)$ has projections with finite volume ratio. Specifically, there exists a subspace $F \subset E$ such that $\dim(F) = \lceil n/4 \rceil$ and

$$v.r.\big(\mathrm{Proj}_F(K)\big) = v.r.\big(\mathrm{Proj}_F\big(\mathrm{Proj}_E(K)\big)\big) < C.$$

Indeed, F is the orthogonal complement inside E, to the subspace G from Lemma 2.2.

We will perform a Schwartz–Steiner symmetrization to the body K, with respect to the subspace F. Denote the resulting body by K'. The body K' is a direct sum of a finite volume ratio body (in the subspace F), and a Euclidean ball (in the subspace $F^{\perp}$). Therefore, also K' is a finite volume ratio body. Since $F \subset E$, we can control the volume of $\mathrm{Proj}_F(K)$ by Claim 3.2:

$$\mathrm{Vol}\big(\mathrm{Proj}_F(K)\big)^{1/n} \leqslant \big(\mathrm{e}^{cn}\,\mathrm{Vol}\big(\mathrm{Proj}_F(\mathcal{E})\big)\big)^{1/n} < c'.$$

Now we can use Lemma 2.3:

$$L_{K'} \approx \frac{L_n^{1/4}}{\mathrm{Vol}(\mathrm{Proj}_F(K))^{1/n}} > cL_n^{1/4}$$

and therefore, $L_n < c(L_0)^4$, where L_0 is the largest possible L_K among all convex bodies which has a volume ratio less than c (where c is some number), and Proposition 1.1 is proved.

Remark. Regarding the connection between v, c_1 and c_2 in Proposition 1.1; Formally, we have proved that $c_2 \lesssim c_1^4$. However, by playing with the dimensions of the subspaces E and F, we can reduce the power of c_1, at the expense of increasing the volume ratio constant. The dependence we get in this way is quite poor: let $L(a) = \sup\{L_K;\, v.r.(K) \leqslant a\}$. Then for any $0 < \theta < 1$,

$$L_n \leqslant \mathrm{e}^{c/(1-\theta)} L\big(\mathrm{e}^{c/(1-\theta)}\big)^{1/\theta}.$$

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