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Group Theory

On quiver varieties and affine Grassmannians of type A

Sur les variétés carquois et grassmannienes affines de type A

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Abstract

We construct Nakajima's quiver varieties of type A in terms of affine Grassmannians of type A. This gives a compactification of quiver varieties and a decomposition of affine Grassmannians into a disjoint union of quiver varieties. Consequently, singularities of quiver varieties, nilpotent orbits and affine Grassmannians are the same in type A. The construction also provides a geometric framework for skew $(\mathrm{GL}(m), \mathrm{GL}(n))$ duality and identifies the natural basis of weight spaces in Nakajima's construction with the natural basis of multiplicity spaces in tensor products which arises from affine Grassmannians. ***To cite this article: I. Mirković, M. Vybornov, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous construisons les variétés carquois de Nakajima de type A en termes de Grassmanniennes affines de type A. Ceci fournit une compactification de ces variétés carquois et une décomposition de ces Grassmanniennes affines en une union disjointe de variétés carquois. En conséquence, les singularités des variétés carquois, des orbites nilpotentes et des Grassmanniennes affines sont les mêmes en type A. La construction fournit aussi un cadre géometrique pour la dualité $(\mathrm{GL}(m), \mathrm{GL}(n))$ extérieure et permet d'identifier la base naturelle des espaces de poids dans la construction de Nakajima avec la base naturelle des espaces de multiplicité des produits tensoriels dans la construction géométrique en termes de Grassmanienne affine. ***Pour citer cet article : I. Mirković, M. Vybornov, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

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Soient $v = (v_1, \ldots, v_{n-1})$ et $d = (d_1, \ldots, d_{n-1})$ deux suites de $n - 1$ entiers positifs ou nuls. Considérons les variétés carquois $\mathfrak{M}(v, d)$ et $\mathfrak{M}_0(v, d)$ de type A_{n-1} définies par Nakajima [12,13], et le morphisme projectif $p : \mathfrak{M}(v, d) \to \mathfrak{M}_0(v, d)$. Notons $\mathfrak{L}(v, d) = p^{-1}(0) \subset \mathfrak{M}(v, d)$. Comme Maffei [10], notons aussi $\mathfrak{M}_1(v, d) = p(\mathfrak{M}(v, d)) \subset \mathfrak{M}_0(v, d)$. A v et d correspondent des poids d et $d - Cv$ de $\mathrm{SL}(n)$, ainsi que des poids (partitions) $\check{\lambda}$ et a de $\mathrm{GL}(n)$, cf. [12, 8.2, 8.3]. Enfin, la dualité usuelle des partitions fournit des copoids λ et μ de $\mathrm{GL}(m)$, où $m = d_1 + \cdots + d_{n-1}$.

Soient $G = \mathrm{GL}(m)$ et $\mathcal{G}_G = G((z))/G[[z]]$ la Grassmanniene affine de G. Notons $\mathcal{G}_\mu$ l'orbite de $G[[z]]$ passant par le copoids dominant (partitions) μ. Soit $\check{\mu}$ la partition duale de μ et $a = (a_1, \ldots, a_n)$ une permutation de $\check{\mu}$. Considérons l'espace de convolution $\widetilde{\mathcal{G}}^a_\mu = \mathcal{G}_{\omega_{a_1}} * \cdots * \mathcal{G}_{\omega_{a_n}}$, où ω_k est le k-ième copoids fondamental de $\mathrm{GL}(m)$, et l'application $\pi = \pi^a_\mu : \widetilde{\mathcal{G}}^a_\mu \to \overline{\mathcal{G}}_\mu$, cf. [11]. Notons $L^{<0}G$ le noyau du morphisme d'évaluation $G[z^{-1}] \to G$, $z^{-1} \mapsto 0$, et notons T_λ l'orbite de $L^{<0}G$ passant par λ dans $\mathcal{G}_G$, où λ est un copoids dominant de G.

Théorème. *Soient v, d, a, λ, μ comme ci-dessus. Il existe des isomorphismes algébriques ϕ et $\tilde{\phi}$, vérifiant $\phi(0) = \lambda$, et tels que le diagramme suivant soit commutatif :*

$$\begin{array}{ccccc}
\mathfrak{M}(v,d) & \xrightarrow[\tilde{\phi}]{\simeq} & \pi^{-1}(T_\lambda \cap \overline{\mathcal{G}}_\mu) & \xrightarrow{\subseteq} & \widetilde{\mathcal{G}}^a_\mu \\
{\scriptstyle p}\downarrow & & {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\
\mathfrak{M}_1(v,d) & \xrightarrow[\phi]{\simeq} & T_\lambda \cap \overline{\mathcal{G}}_\mu & \xrightarrow{\subseteq} & \overline{\mathcal{G}}_\mu
\end{array}$$

En particulier, $\tilde{\phi}$ se restreint en un isomorphisme $\tilde{\phi} : \mathfrak{L}(v, d) \simeq \pi^{-1}(\lambda)$.

1. Preliminaries

1.1. Quiver varieties of type A

We recall Nakajima's construction of simple representations of $\mathrm{SL}(n)$, cf. [12,13]. Let $I = \{1, \ldots, n-1\}$ be the set of vertices and let Ω be an orientation of the Dynkin graph of type A_{n-1}. Let $\overline{\Omega}$ be the opposite orientation and $H \stackrel{\mathrm{def}}{=} \Omega \sqcup \overline{\Omega}$ be the set of *arrows*. We call (I, H) the Dynkin quiver of type A_{n-1}. For an arrow $h \in H$ we denote by $h' \in I$ and $h'' \in I$ its initial and terminal vertices. For a pair v, d in $\mathbb{Z}^I_{\geqslant 0}$ take $\mathbb{C}$-vector spaces V_i and D_i of dimensions $\dim V_i = v_i$ and $\dim D_i = d_i$, $i \in I$. Consider the affine space

$$M(v, d) = \bigoplus_{h \in H} \mathrm{Hom}(V_{h'}, V_{h''}) \oplus \bigoplus_{i \in I} \mathrm{Hom}(D_i, V_i) \oplus \bigoplus_{i \in I} \mathrm{Hom}(V_i, D_i)$$

with the natural action of the group $G(V) = \prod_{i \in I} \mathrm{GL}(V_i)$. Let $\mathbf{m} : M(v, d) \to \mathfrak{g}(V)$ be the corresponding moment map into the Lie algebra $\mathfrak{g}(V)$. (Symplectic form on $M(v, d)$ is defined in [13, 3], the Lie algebra $\mathfrak{g}(V)$ is identified with its dual via the trace.) Denote $\Lambda(v, d) = \mathbf{m}^{-1}(0)$.

Nakajima's quiver variety $\mathfrak{M}(v, d)$ is the geometric quotient of $\Lambda^s(v, d)$ by $G(V)$, where $\Lambda^s(v, d)$ is the set of all stable points in $\Lambda(v, d)$ (so $\Lambda^s(v, d)/G(V)$ is the set of $\mathbb{C}$-points of $\mathfrak{M}(v, d)$). The quiver variety $\mathfrak{M}_0(v, d) = \Lambda(v, d)//G(V)$ is the invariant theory quotient (the spectrum of the $G(V)$-invariant functions). There is a natural projective map $p : \mathfrak{M}(v, d) \to \mathfrak{M}_0(v, d)$, cf. [13], and following Maffei [10], denote its image by

$\mathfrak{M}_1(v,d) = p(\mathfrak{M}(v,d)) \subseteq \mathfrak{M}_0(v,d)$. Finally, let $\mathfrak{L}(v,d) \stackrel{\text{def}}{=} p^{-1}(0) \subseteq \mathfrak{M}(v,d)$ and denote by $\mathcal{H}(\mathfrak{L}(v,d))$ its top-dimensional Borel–Moore homology.

Theorem 1.1 [13, 10.ii]. *The space $\bigoplus_v \mathcal{H}(\mathfrak{L}(v,d))$ has the structure of a simple* $\mathrm{SL}(n)$*-module with the highest weight d (i.e., $\sum_I d_i\omega_i$ for the fundamental weights ω_i). The summand $\mathcal{H}(\mathfrak{L}(v,d))$ is the weight space for the weight $v' = \sum_I v'_i\omega_i = \sum_I d_i\omega_i - \sum_I v_i\alpha_i$, where $\{\alpha_i,\ i \in I\}$ are simple roots of* $\mathrm{SL}(n)$.

1.2. *From* SL(n) *to* GL(n)

We may consider $\bigoplus_v \mathcal{H}(\mathfrak{L}(v,d))$ as a representation $W_{\check\lambda}$ of $\mathrm{GL}(n)$ with the highest weight $\check\lambda = \check\lambda(d) = (\check\lambda_1, \check\lambda_2, \ldots, \check\lambda_n)$, where $\check\lambda_i = \sum_{j=i}^{n-1} d_j$ for $1 \leqslant i \leqslant n-1$ and $\check\lambda_n = 0$, is a partition of $N = \sum_{j=1}^{n-1} j d_j$. Then $\mathcal{H}(\mathfrak{L}(v,d))$ is the weight space $W_{\check\lambda}(a)$, where $a_i = v_{n-1} + \sum_{j=i}^{n} v'_j$ (here $v'_n = 0$), cf. [12, 8.3].

1.3. *Affine Grassmannians of type A*

We recall the construction of representations of $G = \mathrm{GL}(m)$ in terms of its affine Grassmannian $\mathcal{G}_G$, cf. [8, 5,11]. Let V be a vector space with a basis $\{e_1, \ldots, e_m\}$ and $V((z)) = V \otimes_{\mathbb{C}} \mathbb{C}((z)) \supseteq L_0 = V \otimes_{\mathbb{C}} \mathbb{C}[[z]]$. A *lattice* in $V((z))$ is an $\mathbb{C}[[z]]$-submodule L of $V((z))$ such that $L \otimes_{\mathbb{C}[[z]]} \mathbb{C}((z)) = V((z))$. The affine Grassmannian $\mathcal{G}_G$ is a reduced ind-scheme whose $\mathbb{C}$-points can be described as all lattices in $V((z))$ or as $G((z))/G[[z]]$. Its connected components $\mathcal{G}_{(N)}$ are indexed by integers $N \in \mathbb{Z}$, and if $N \geqslant 0$ then $\mathcal{G}_{(N)}$ contains $\mathcal{G}_N = \{\text{lattices } L \text{ in } V((z)) \text{ such that } L_0 \subseteq L,\ \dim L/L_0 = N\}$. To a dominant coweight $\lambda \in \mathbb{Z}^m$ of G, one attaches the lattice $L_\lambda = \bigoplus_1^m \mathbb{C}[[z]] \cdot z^{-\lambda_i} e_i$. The $G[[z]]$-orbits $\mathcal{G}_\lambda$ in $\mathcal{G}_G$ are parameterized by the dominant coweights λ via $\mathcal{G}_\lambda = G[[z]] \cdot L_\lambda$. Finally, we denote by $L^{<0}G$ the subgroup of the group ind-scheme $G[z^{-1}]$ defined as the kernel of the evaluation $z^{-1} \mapsto 0$.

The intersection homology of the closure $\overline{\mathcal{G}}_\lambda$ is a realization of the representation V_λ, and the convolution of IC-sheaves corresponds to the tensor products of representations, cf. [5,11].

1.4. *Resolution of singularities*

The closure $\overline{\mathcal{G}}_\mu$ of the orbit $\mathcal{G}_\mu$ in $\mathcal{G}_N$ has a natural resolution. The $G[[z]]$-orbits in $\mathcal{G}_N$ correspond to μ's which may be considered as partitions of N (into at most m parts). Any permutation $a = (a_1, \ldots, a_n)$ of the partition $\check\mu$ dual to μ defines a convolution space $\widetilde{\mathcal{G}}^a_\mu = \mathcal{G}_{\omega_{a_1}} * \cdots * \mathcal{G}_{\omega_{a_n}}$, where ω_k is the k-th fundamental coweight of G, and a resolution of singularities $\pi = \pi^a_\mu : \widetilde{\mathcal{G}}^a_\mu \to \overline{\mathcal{G}}_\mu$, cf. [11].

2. Nilpotent cones of type A

2.1. *n-flags* [4,3]

Let us fix a vector space D of dimension N. Let $\mathcal{N} = \mathcal{N}(D)$ be the nilpotent cone in $\mathrm{End}(D)$. The connected components $\mathcal{F}^{n,a}$ of the variety of n-step flags in D are parameterized by all $a \in \mathbb{Z}^n_{\geqslant 0}$ such that $N = \sum_{i=1}^n a_i$:

$$\mathcal{F}^{n,a} = \bigl\{0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = D \mid \dim F_i - \dim F_{i-1} = a_i\bigr\}.$$

Its cotangent bundle is $\widetilde{\mathcal{N}}^{n,a} = T^*\mathcal{F}^{n,a} = \{(u,F) \in \mathcal{N} \times \mathcal{F}^a \mid u(F_i) \subseteq F_{i-1}\}$. Denote by $\mathbf{m}_a : \widetilde{\mathcal{N}}^{n,a} \to \mathcal{N}$ the projection onto the first factor.

2.2. A transverse slice to a nilpotent orbit

Let x be a nilpotent operator on D, with Jordan blocks of sizes $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m)$. We construct a "transverse slice" T_x to the nilpotent orbit $\mathcal{O}_\lambda \subseteq \mathcal{N}$ at x, different from the one considered by Slodowy [14, 7.4]. In some basis $e_{k,i}$, $1 \leqslant k \leqslant \lambda_i$, of D, one has $x : e_{k,i} \mapsto e_{k-1,i}$ (we set $e_{0,i} = 0$). Now

$$T_x \stackrel{\text{def}}{=} \left\{x + f,\ f \in \operatorname{End}(D) \mid f_{k,i}^{l,j} = 0,\ \text{if } k \neq \lambda_i,\ \text{and } f_{\lambda_i,i}^{l,j} = 0,\ \text{if } l > \lambda_i\right\},$$

where $f_{k,i}^{l,j} : \mathbb{C}e_{l,j} \to \mathbb{C}e_{k,i}$ are the matrix elements of f in our basis. For a larger orbit $\mathcal{O}_\mu$, any permutation $a = (a_1, \ldots, a_n)$ of the dual partition $\check{\mu}$, gives a resolution $\mathcal{T}_x^a \stackrel{\text{def}}{=} \mathbf{m}_a^{-1}(T_x \cap \overline{\mathcal{O}}_\mu) \subset \widetilde{\mathcal{N}}^{n,a}$ of the slice $T_{x,\mu} \stackrel{\text{def}}{=} T_x \cap \overline{\mathcal{O}}_\mu$ to $\mathcal{O}_\lambda$ in $\overline{\mathcal{O}}_\mu$.

3. Main theorem

3.1. From quiver data to affine Grassmannian data

We start with A_{n-1} quiver data $v, d \in \mathbb{Z}_{\geqslant 0}^I$ such that $\mathfrak{M}(v,d)$ is nonempty. Take the $\mathrm{SL}(n)$-weights d and $d - Cv$, and pass to $\mathrm{GL}(n)$-weights $\check{\lambda}$ and a as in Section 1.2. Now permute a to a partition $\check{\mu} = \check{\mu}(a) = (\check{\mu}_1 \geqslant \check{\mu}_2 \geqslant \cdots \geqslant \check{\mu}_n)$ of $N = \sum_{j=1}^{n-1} j d_j$. Finally, let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$, where $m = \sum_{i=1}^{n-1} d_i$, be the partitions of N (i.e., $\mathrm{GL}(m)$-coweights) dual to $\check{\lambda}$ and $\check{\mu}$ respectively.

Theorem 3.1. *Let N, v, d, a, λ, μ be as above. Let $L_\lambda \in \mathcal{G}_G$ be the lattice corresponding to the coweight λ, and let T_λ be its $L^{<0}G$-orbit. There exist algebraic isomorphisms ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ such that the following diagram commutes*:

$$\begin{array}{ccccccc}
\mathfrak{M}(v,d) & \xrightarrow[\simeq]{\tilde{\phi}} & \widetilde{T}_x^a & \xrightarrow[\simeq]{\tilde{\psi}} & \pi^{-1}(T_\lambda \cap \overline{\mathcal{G}}_\mu) & \xrightarrow{\subseteq} & \widetilde{\mathcal{G}}_\mu^a \\
\Big\downarrow p & & \Big\downarrow \mathbf{m}_a & & \Big\downarrow \pi & & \Big\downarrow \pi \\
\mathfrak{M}_1(v,d) & \xrightarrow[\simeq]{\phi} & T_{x,\mu} & \xrightarrow[\simeq]{\psi} & T_\lambda \cap \overline{\mathcal{G}}_\mu & \xrightarrow{\subseteq} & \overline{\mathcal{G}}_\mu
\end{array} \tag{1}$$

and $(\psi \circ \phi)(0) = L_\lambda$. In particular, $\tilde{\psi} \circ \tilde{\phi}$ restricts to an isomorphism $\mathfrak{L}(v,d) \simeq \pi^{-1}(L_\lambda)$.

3.2.

For $d = (d_1, 0, \ldots, 0)$ and $\lambda = (1, \ldots, 1)$ the theorem above was proven in (or follows immediately from) [8,12]. The isomorphisms ϕ (resp. $\tilde{\phi}$) is analogous to the isomorphism constructed in [12] (resp. isomorphism conjectured in [12, 8.6] and constructed in [10] using a result from [9]). However, our isomorphism ϕ is given by an explicit formula described as follows. Let us think of a point in $\mathfrak{M}_1(v,d)$ as (closed orbit of) a quadruple $(\{B_i\}_{i\in I}, \{\overline{B}_i\}_{i\in I}, \{p_i\}_{i\in I}, \{q_i\}_{i\in I}) \in \Lambda(v,d)$, where $B_i \in \operatorname{Hom}(V_i, V_{i+1})$, $\overline{B}_i \in \operatorname{Hom}(V_{i+1}, V_i)$, $p_i \in \operatorname{Hom}(D_i, V_i)$, and $q_i \in \operatorname{Hom}(V_i, D_i)$. We decompose the vector space D, $\dim D = N$, as a direct sum: $D = \bigoplus_{1 \leqslant h \leqslant j \leqslant n-1} D_j^h$, cf. [10], where $D_j^h = \mathbb{C}\{e_{h,i} \mid \lambda_i = j\}$, $\dim D_j^h = \dim D_j = d_j$ (notation of 1.1, 2.2, 3.1). Then for any $f \in \operatorname{End}(D)$ we consider its blocks $f_{j,h}^{j',h'} : D_{j'}^{h'} \to D_j^h$. By definition, $\phi(B, \overline{B}, p, q) = x + f \in \mathcal{N}$ (notation of 2.2), where

$$f_{j,h}^{j',h'} = \begin{cases} q_j B_{j-1} \ldots B_{h'+1} B_{h'} \overline{B}_{h'} \overline{B}_{h'+1} \ldots \overline{B}_{j'-1} p_{j'} & \text{if } h = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In particular, $\phi(0) = x$.

3.3. Compactification of quiver varieties

A compactification of $\mathfrak{M}_1(v,d)$ and $\mathfrak{M}(v,d)$ is given by closures of their respective images under the embeddings $\mathfrak{M}_1(v,d) \hookrightarrow \overline{\mathcal{G}}_\mu$ and $\mathfrak{M}(v,d) \hookrightarrow \widetilde{\mathcal{G}}^a_\mu$.

3.4. Decomposition

The theorem implies a decomposition of $\overline{\mathcal{G}}_\mu$ into a disjoint union of quiver varieties

$$\overline{\mathcal{G}}_\mu = \bigsqcup_{\mathcal{G}_\lambda \subseteq \overline{\mathcal{G}}_\mu} \bigsqcup_{y \in G \cdot L_\lambda} \mathfrak{M}_0(v,d)_y, \tag{3}$$

where $\mathfrak{M}_0(v,d)_y$ is a copy of quiver variety $\mathfrak{M}_0(v,d)$ for every point $y \in G \cdot L_\lambda$, and v, d are obtained from λ, μ by reversing formulas in Section 1.2.

3.5. Beilinson–Drinfeld Grassmannians

Recall the moment map $\mathbf{m} : M(v,d) \to \mathfrak{g}(V)$ from Section 1.1. Any $c = (c_1 \operatorname{Id}_{V_1}, \ldots, c_{n-1} \operatorname{Id}_{V_{n-1}})$ in the center of the Lie algebra $\mathfrak{g}(V)$ defines $\Lambda_c(v,d) = \mathbf{m}^{-1}(c)$, and then, as in 1.1, the "deformed" quiver varieties $\mathfrak{M}^c(v,d) = \Lambda^s_c(v,d)/G(V)$ and $\mathfrak{M}^c_0(v,d) = \Lambda_c(v,d)//G(V)$. We expect that in type A our theorem and decomposition (3) extend to a relation between deformed quiver varieties and the Beilinson–Drinfeld Grassmannians, cf. [2].

For instance, when $d = (d_1, 0, \ldots, 0)$ there is an embedding $\mathfrak{M}^c_0(v,d) \hookrightarrow \mathcal{G}^{\mathrm{BD}}_{\mathbb{A}^{(n)}}(\mathrm{GL}(m))$ of our quiver variety into the fiber of the Beilinson–Drinfeld Grassmannian over the point $(0, c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{n-1}) \in \mathbb{A}^{(n)}$.

The proofs and more details will appear in a forthcoming paper.

Another example of a decomposition of an infinite Grassmannian into a disjoint union of quiver varieties can be found in [1] (who generalized a result from [15]). A part of adelic Grassmannian is a union of quiver varieties $\mathfrak{M}^c(v,d)$ associated to *affine* quivers of type A.

4. Geometric construction of skew $(\mathrm{GL}(n), \mathrm{GL}(m))$ duality

4.1. Skew duality

Let $V = \mathbb{C}^m$ and $W = \mathbb{C}^n$ be two vector spaces. Then we have the $\mathrm{GL}(V) \times \mathrm{GL}(W)$-decomposition, see, e.g., [6, 4.1.1], cf. [7]:

$$\wedge^N (V \otimes W) = \bigoplus_\lambda V_\lambda \otimes W_{\check{\lambda}}, \tag{4}$$

where λ varies over all partitions of N which fit into the $n \times m$ box, and V_λ and $W_{\check{\lambda}}$ are the corresponding highest weight representation of $\mathrm{GL}(m)$ of $\mathrm{GL}(n)$. This is essentially equivalent to natural isomorphisms of vector spaces

$$\operatorname{Hom}_{\mathrm{GL}(m)}\left(\wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V, V_\lambda\right) \simeq W_{\check{\lambda}}(a), \tag{5}$$

where $W_{\check{\lambda}}(a)$ is the weight space corresponding to the weight $a = (a_1, \ldots, a_n)$.

4.2.

We construct a based version of the isomorphism (5), i.e., a geometric skew duality. More precisely, with N, v, d, a, λ as in 3.1, we identify the left-hand side with $\mathcal{H}(\pi^{-1}(L_\lambda))$ (notation from Theorem 3.1) and the right-hand side with $\mathcal{H}(\mathfrak{L}(v,d))$ by Theorem 1.1. The identification of irreducible components $\operatorname{Irr} \pi^{-1}(L_\lambda) = \operatorname{Irr} \mathfrak{L}(v,d)$, which follows from Theorem 3.1, matches the natural basis of the space of intertwiners $\mathrm{Hom}_{\mathrm{GL}(m)}(\wedge^{a_1}V\otimes\cdots\otimes\wedge^{a_n}V, V_\lambda)$ arising from the affine Grassmannian construction (i.e., $\mathrm{Irr}\,\pi^{-1}(L_\lambda)$), and the natural basis of the weight space $W_{\check\lambda}(a)$ in the Nakajima construction (i.e., $\mathrm{Irr}\,\mathfrak{L}(v,d)$). Altogether:

$$\mathrm{Hom}_{\mathrm{GL}(m)}\big(\wedge^{a_1}V\otimes\cdots\otimes\wedge^{a_n}V, V_\lambda\big)\simeq\mathcal{H}\big(\pi^{-1}(L_\lambda)\big)\simeq\mathcal{H}\big(\mathfrak{L}(v,d)\big)\simeq W_{\check\lambda}(a).$$

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