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Partial Differential Equations

A fully nonlinear version of the Yamabe problem and a Harnack type inequality

Une version complètement nonlinéaire du problème de Yamabe et une inégalité du type Harnack

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Abstract

We present some results on a fully nonlinear version of the Yamabe problem and a Harnack type inequality for general conformally invariant fully nonlinear second order elliptic equations. ***To cite this article: A. Li, Y.Y. Li, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

On étudie une version complètement nonlinéaire du problème de Yamabe. On etablit aussi une inégalité du type Harnack pour des équations elliptiques de second ordre, complètement nonlinéaires, avec invariance conforme. Les démonstrations détaillées de ces résultats sont présentées ailleurs. ***Pour citer cet article : A. Li, Y.Y. Li, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Étant donnée une variété riemannienne (M, g) de dimension $n \geqslant 3$, on considère le tenseur de Weyl–Schouten $A_g = \frac{1}{n-2}(\mathrm{Ric}_g - \frac{R_g}{2(n-1)} g)$, où Ric_g et R_g sont la courbure de Ricci et la courbure scalaire associées à g. On dénote par $\lambda(A_g)$ les valeurs propres de A_g par rapport à g. Soit $\Gamma \subset \mathbb{R}^n$ un cône ouvert convexe dont le sommet est à l'origine, symétrique par rapport à λ_i, tel que $\{\lambda \in \mathbb{R}^n \mid \lambda_i > 0, 1 \leqslant i \leqslant n\} \subset \Gamma \subset \{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 0\}$. Soit $f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ une fonction concave et symétrique par rapport à λ_i telle que $f = 0$ sur $\partial\Gamma$, $f_{\lambda_i} > 0$ sur Γ pour chaque $1 \leqslant i \leqslant n$ et $\lim_{s\to\infty} f(s\lambda) = \infty$ pour tout $\lambda \in \Gamma$. Nous avons obtenu les résultats suivants

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Théorème 1 [9]. *Soit $n \geqslant 3$. On suppose que (f, Γ) vérifie les propriétés ci-dessus. Étant donnée une variété riemannienne régulière sans bord (M, g) de dimension n et localement conformément plate telle que $\lambda(A_g) \in \Gamma$ sur M, il existe une fonction régulière u définie sur M telle que $\hat{g} = u^{4/(n-2)} g$ vérifie $f(\lambda(A_{\hat{g}})) = 1$, $\lambda(A_{\hat{g}}) \in \Gamma$, sur M. De plus, si (M, g) n'est pas conformément difféomorphe à la sphère euclidienne de dimension n, alors il existe un entier positif m et une constante C (qui dépend seulement de (M, g), (f, Γ) et m) tels que $\|u\|_{C^m(M,g)} + \|u^{-1}\|_{C^m(M,g)} \leqslant C$ pour toute solution u du problème.*

Théorème 2 [9]. *Soit $n \geqslant 3$. On suppose que $U \subset \mathcal{S}^{n\times n}$ vérifie* (13) *et* (14) *et que $F \in C^1(U)$ vérifie* (12), (15) *et* (16). *Pour chaque $R > 0$, on dénote par B_{3R} la boule de rayon $3R$ dans $\mathbb{R}^n$. Si $u \in C^2(B_{3R})$ est une solution positive de Éq.* (17), *alors u vérifie* (18) *pour une certaine constante $C(n)$ qui peut être calculée explicitément.*

We present some results in [9], a continuation of our earlier works [7,8]. Let (M, g) be an n-dimensional, compact, smooth Riemannian manifold without boundary, $n \geqslant 3$, consider the Weyl–Schouten tensor $A_g = \frac{1}{n-2}(\mathrm{Ric}_g - \frac{R_g}{2(n-1)} g)$, where Ric_g and R_g denote respectively the Ricci tensor and the scalar curvature associated with g. We use $\lambda(A_g)$ to denote the eigenvalues of A_g. Let $\hat{g} = u^{4/(n-2)} g$, then (see, e.g., [17]),

$$A_{\hat{g}} = -2(n-2)^{-1} u^{-1} \nabla_g^2 u + 2n(n-2)^{-2} u^{-2} \nabla_g u \otimes \nabla_g u - 2(n-2)^{-2} u^{-2} |\nabla_g u|_g^2 g + A_g. \tag{1}$$

Let

$$\Gamma \subset \mathbb{R}^n \text{ be an open convex cone with vertex at the origin,} \tag{2}$$

$$\bigl\{\lambda \in \mathbb{R}^n \mid \lambda_i > 0, 1 \leqslant i \leqslant n\bigr\} \subset \Gamma \subset \bigl\{\lambda \in \mathbb{R}^n \mid \lambda_1 + \cdots + \lambda_n > 0\bigr\}, \tag{3}$$

$$\Gamma \text{ is symmetric in the } \lambda_i, \tag{4}$$

$$f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma}) \text{ be concave and symmetric in the } \lambda_i, \tag{5}$$

$$f = 0 \quad \text{on } \partial\Gamma; \qquad f_{\lambda_i} > 0 \quad \text{on } \Gamma,\ \forall 1 \leqslant i \leqslant n, \tag{6}$$

$$\lim_{s\to\infty} f(s\lambda) = \infty, \quad \forall \lambda \in \Gamma. \tag{7}$$

Theorem 1 [9]. *For $n \geqslant 3$, let (f, Γ) satisfy* (2)–(7) *and let (M, g) be an n-dimensional smooth compact locally conformally flat Riemannian manifold without boundary satisfying*

$$\lambda(A_g) \in \Gamma, \quad \text{on } M. \tag{8}$$

Then there exists some smooth positive function u on M such that $\hat{g} = u^{4/(n-2)} g$ satisfies

$$f\bigl(\lambda(A_{\hat{g}})\bigr) = 1, \quad \lambda(A_{\hat{g}}) \in \Gamma \text{ on } M. \tag{9}$$

Moreover, if (M, g) is not conformally diffeomorphic to the standard n-sphere, all solutions of the above satisfy, for any positive integer m, and some constant C depending only on (M, g), (f, Γ) and m,

$$\|u\|_{C^m(M,g)} + \bigl\|u^{-1}\bigr\|_{C^m(M,g)} \leqslant C. \tag{10}$$

Remark 1. The C^0 and C^1 apriori estimates above do not require the concavity of f.

For $1 \leqslant k \leqslant n$, let $\sigma_k(\lambda) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}$, denote the k-th symmetric function, and let $\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0\}$. Then (see [2]) $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ satisfies the hypothesis in Theorem 1. For $(f, \Gamma) = (\sigma_1, \Gamma_1)$, hypothesis (8) is equivalent to $R_g > 0$ on M, and Theorem 1 in this case is the Yamabe problem for locally conformally flat manifolds with positive Yamabe invariants, and the result is due to Schoen [12, 13]. The Yamabe conjecture was proved through the work of Yamabe, Trudinger, Aubin and Schoen. For $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ with $k = 2$ and $n = 4$, the result was proved without the locally conformally flatness hypothesis of the manifold by Chang, Gursky and Yang [3]. For $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$ with $k=n\geqslant 3$, some existence result was established by Viaclovsky [16] for a class of manifolds which are not necessarily locally conformally flat. For $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$, $n\geqslant 3$, $1\leqslant k\leqslant n$, Theorem 1 was established in [7,8]; while the existence part in the case $k\neq\frac{n}{2}$ was independently obtained by Guan and Wang in [5]. Subsequently, Guan, Viaclovsky and Wang [4] proved the algebraic fact that $\lambda(A_g)\in\Gamma_k$ for $k\geqslant\frac{n}{2}$ implies the positivity of the Ricci tensor, and therefore both the existence and compactness results in this case follow from known results. More recently, Gursky and Viaclovsky [6] have obtained existence results for $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$, $n=3,4$, on general Riemannian manifolds.

A Liouville type theorem for $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$ was established in [8]. The crucial ingredient in our proof of the Liouville type theorem is a Harnack type inequality for $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$ established in the same paper. In [9], we have established the Harnack type inequality for general conformally invariant fully nonlinear second order elliptic equations. In the following, $\mathcal{S}^{n\times n}$ denotes the set of $n\times n$ real symmetric matrices, $\mathcal{S}_+^{n\times n}\subset\mathcal{S}^{n\times n}$ denotes the set of positive definite matrices, $\mathrm{O}(n)$ denotes the set of $n\times n$ real orthogonal matrices, and I denotes the $n\times n$ identity matrix. It was show in [8] that $H(\cdot,u,\nabla u,\nabla^2u)$ is conformally invariant on $\mathbb{R}^n$ (see [8] for the definition) if and only if $H(\cdot,u,\nabla u,\nabla^2u)\equiv F(A^u)$, where

$$\begin{aligned}A^u &:= -2(n-2)^{-1}u^{-(n+2)/(n-2)}\nabla^2u+2n(n-2)^{-2}u^{-2n/(n-2)}\nabla u\otimes\nabla u\\ &\quad -2(n-2)^{-2}u^{-2n(n-2)}|\nabla u|^2I, \qquad (11)\end{aligned}$$

$$F\big(O^{-1}MO\big)=F(M),\quad \forall M\in\mathcal{S}^{n\times n},\ \forall O\in\mathrm{O}(n). \qquad (12)$$

Let $U\subset\mathcal{S}^{n\times n}$ be an open set satisfying

$$O^{-1}UO=U,\quad \forall O\in\mathrm{O}(n), \qquad (13)$$

$$U\cap\{M+tN\mid 0<t<\infty\}\text{ is convex }\forall M\in\mathcal{S}^{n\times n},\ N\in\mathcal{S}_+^{n\times n}. \qquad (14)$$

Let $F\in C^1(U)$ satisfy (12) and

$$\left(\frac{\partial F}{\partial M_{ij}}(M)\right)>0,\quad \forall M\in U, \qquad (15)$$

$$F(M)\neq 1,\quad \forall M\in U\cap\left\{M\in\mathcal{S}^{n\times n}\mid \|M\|:=\left(\sum_{i,j}M_{ij}^2\right)^{1/2}<\delta\right\}. \qquad (16)$$

For $F_k(M):=\sigma_k^{1/k}(\lambda(M))$, and $U_k:=\{M\in\mathcal{S}^{n\times n}\mid\lambda(M)\in\Gamma_k\}$, it is well known that $(F,U)=(F_k,U_k)$ satisfies (13), (14), (12) and (16).

Theorem 2 [9]. *For $n\geqslant 3$, let $U\subset\mathcal{S}^{n\times n}$ satisfy* (13) *and* (14)*, and let $F\in C^1(U)$ satisfy* (12), (15) *and* (16). *For $R>0$, let B_R be a ball in $\mathbb{R}^n$ of radius R, and let $u\in C^2(B_{3R})$ be a positive solution of*

$$F\big(A^u\big)=1,\quad A^u\in U,\ \text{in } B_{3R}. \qquad (17)$$

Then, for some constant $C(n)$ depending only on n,

$$\Big(\sup_{B_R}u\Big)\Big(\inf_{B_{2R}}u\Big)\leqslant C(n)\delta^{(2-n)/2}R^{2-n}, \qquad (18)$$

Remark 2. In Theorem 2, there is no concavity assumption on F and the constant $C(n)$ can be given explicitly.

Remark 3. The Harnack type inequality (18) for $(F,U)=(F_1,U_1)$ was obtained by Schoen in [14] based on a Liouville type theorem of Caffarelli, Gidas and Spruck in [1]. Li and Zhang gave in [11] a different proof of Schoen's Harnack type inequality without using the Liouville type theorem. For $(F,U)=(F_k,U_k)$, $1\leqslant k\leqslant n$, the Harnack type inequality was established in our earlier work [8]. There are two new ingredients in our proof of Theorem 2. One is that we have developed, along the line of [8], new C^0 and C^1 estimates which allow us to extend the Harnack type inequality in [8] to this generality, and the other is that we have given a direct proof which makes it possible to give an explicit constant C in (18). Arguments in [14,11] and [8] were indirect and therefore no explicit value of C was available, even in the case $(F,U)=(F_1,U_1)$.

We first present our proof of Theorem 1, more details can be found in [9]. As explained in [9], we may further assume without loss of generality that f is homogeneous of degree 1. By (6) and (7), there exists a unique $b>0$ such that $f(be)=1$, where $e=(1,\dots,1)$. By (6), there exists some $\delta_1>0$ such that

$$f(\lambda)<1,\quad \forall\lambda\in\Gamma,\ |\lambda|<\delta_1. \tag{19}$$

Fix some constant δ_2 such that

$$0<\delta_2\leqslant\min_{x\in M} f\big(\lambda\big(A_g(x)\big)\big). \tag{20}$$

Let $(\widetilde M,\tilde g)$ denote the universal cover of (M,g), with $i:\widetilde M\to M$ a covering map and $\tilde g=i^*g$. By a theorem of Schoen and Yau in [15], there exists an injective conformal immersion $\Phi:(\widetilde M,\tilde g)\to(\mathbb{S}^n,g_0)$, where g_0 denotes the standard metric on $\mathbb{S}^n$. Moreover, $\Omega:=\Phi(\widetilde M)$ is either $\mathbb{S}^n$ or an open and dense subset of $\mathbb{S}^n$. Fix a compact subset E of $\widetilde M$ such that $i(E)=M$. To prove Theorem 1, we will establish (10) first. Let $u\in C^\infty(M)$ be a positive solution of (9) with $\hat g=u^{4/(n-2)}g$. We denote $F(A_g):=f(\lambda(A_g))$.

Step 1. For some constant C depending only on (M,g), b, δ_1 and δ_2, we have

$$C^{-1}\leqslant u\leqslant C,\quad |\nabla_g u|\leqslant C\quad\text{on } M. \tag{21}$$

Two cases: *Case* 1. $\Omega=\mathbb{S}^n$; *Case* 2. $\Omega\neq\mathbb{S}^n$.

In Case 1, $(\Phi^{-1})^*\tilde g=\eta^{4/(n-2)}g_0$ on $\mathbb{S}^n$, where η is a positive smooth function on $\mathbb{S}^n$. Let $\tilde u=u\circ i$. Since $F(A_{\tilde u^{4/(n-2)}\tilde g})=1$ on $\widetilde M$, we have $F(A_{[(\tilde u\circ\Phi^{-1})\eta]^{4/(n-2)}g_0})=1$, on $\mathbb{S}^n$. By Corollary 1.1 in [8], $(\tilde u\circ\Phi^{-1})\eta=a|J_\varphi|^{(n-2)/(2n)}$ for some positive constant a and some conformal diffeomorphism φ of $\mathbb{S}^n$. Since $\varphi^*g_0=|J_\varphi|^{2/n}g_0$, we have $f(a^{-4/(n-2)}(n-1)e)=f(a^{-4/(n-2)}\lambda(A_{g_0}))=1$, i.e., $(n-1)a^{-4/(n-2)}=b$. Estimate (10) follows easily.

In Case 2, $(\Phi^{-1})^*\tilde g=\eta^{4/(n-2)}g_0$ on Ω where, by [15], η is a positive smooth function in Ω satisfying $\lim_{z\to\partial\Omega}\eta(z)=\infty$. Recall that Ω is an open and dense subset of $\mathbb{S}^n$. Let $u(x)=\max_M u$ for some $x\in M$, and let $i(\tilde x)=x$ for some $\tilde x\in E$. By composing with a rotation of $\mathbb{S}^n$, we may assume without loss of generality that $\Phi(\tilde x)=S$, the south pole of $\mathbb{S}^n$. Let $P:\mathbb{S}^n\to\mathbb{R}^n$ be the stereographic projection, and let v be the positive function on the open subset $P(\Omega)$ of $\mathbb{R}^n$ determined by $(P^{-1})^*(\eta^{4/(n-2)}g_0)=v^{4(n-2)}g_{\mathrm{flat}}$, where g_{flat} denotes the Euclidean metric on $\mathbb{R}^n$. Then for some $\varepsilon>0$, depending only on (M,g), we have $B_{9\varepsilon}:=\{x\in\mathbb{R}^n\mid |x|<9\varepsilon\}\subset P(\Omega)$, and $\mathrm{dist}_{g_{\mathrm{flat}}}(P(\Phi(E)),\partial P(\Omega))>9\varepsilon$. Let $\hat u=(\tilde u\circ\Phi^{-1}\circ P^{-1})v$ on $P(\Omega)$, we have, by (1), $f(\lambda(A^{\hat u}))=1$ and $\lambda(A^{\hat u})\in\Gamma$. By the property of η, we know that $\lim_{y\to\bar y,\ y\in P(\Omega)}\hat u(y)=\infty$ for all $\bar y\in\partial P(\Omega)$ and, if the north pole of $\mathbb{S}^n$ does not belong to Ω, $\lim_{y\in P(\Omega),\ |y|\to\infty}(|y|^{n-2}\hat u(y))=\infty$. By a moving sphere argument as in [9], we have, for every $x\in\mathbb{R}^n$ satisfying $\mathrm{dist}_{g_{\mathrm{flat}}}(x,P(\Phi(E)))<2\varepsilon$, that

$$\hat u_{x,\lambda}(y):=\frac{\lambda^{n-2}}{|y-x|^{n-2}}\hat u\left(\frac{\lambda^2(y-x)}{|y-x|^2}\right)\leqslant\hat u(y),\quad \forall 0<\lambda<4\varepsilon,\ |y-x|\geqslant\lambda,\ y\in P(\Omega). \tag{22}$$

Lemma 1 [9]. *Let $a>0$ be a constant and let $B_{8a}\subset\mathbb{R}^n$ be the ball of radius $8a$ and centered at the origin, $n\geqslant 3$. Assume that $u\in C^1(B_{8a})$ is a non-negative function satisfying*

$$u_{x,\lambda}(y)\leqslant u(y),\quad \forall x\in B_{4a},\ y\in B_{8a},\ 0<\lambda<2a,\ \lambda<|y-x|,$$

where $u_{x,\lambda}(y):=(\lambda/|y|)^{n-2}u(x+\lambda^2(y-x)/|y-x|^2)$. Then $|\nabla u(x)|\leqslant((n-2)/(2a))u(x),\ \forall|x|<a$.

By (22) and the above calculus lemma, we have $|\nabla(\log\hat u)(y)|\leqslant C(\varepsilon),\ \forall\,\mathrm{dist}_{g_{\mathrm{flat}}}(y,P(\Phi(E)))<\varepsilon$.

Thus, for some constant C depending only on (M,g), $|\nabla_g \log u| \leqslant C$ on M, and

$$\sup_{B_\varepsilon} \hat{u} \leqslant C \inf_{B_\varepsilon} \hat{u}. \tag{23}$$

Let $\beta > 0$ be the constant such that $\xi(y) := \beta(\varepsilon^2 - |y|^2)$ has the property that $\hat{u} \geqslant \xi$ on B_ε, and, for some $\bar{y} \in B_\varepsilon$, $\hat{u}(\bar{y}) = \xi(\bar{y})$. It follows that $\nabla\hat{u}(\bar{y}) = \nabla\xi(\bar{y})$, $(D^2\hat{u}(\bar{y})) \geqslant (D^2\xi(\bar{y}))$, and $A^{\hat{u}}(\bar{y}) \leqslant A^{\xi}(\bar{y})$. By (23) and the definition of ξ, we have $1-(|\bar{y}|/\varepsilon)^2 \geqslant C^{-1}$, and $C^{-1}\sup_{B_\varepsilon}\hat{u} \leqslant \beta\varepsilon^2 \leqslant C\inf_{B_\varepsilon}\hat{u}$, where C depends only on (M,g). Consequently, $A^{\hat{u}}(\bar{y}) \leqslant A^{\xi}(\bar{y}) \leqslant C\beta^{-4/(n-2)}I$. This, together with the fact that $\lambda(A^{\hat{u}}(\bar{y})) \in \Gamma \subset \Gamma_1$, implies that $|\lambda(A^{\hat{u}}(\bar{y}))| \leqslant C\beta^{-4/(n-2)}$. Since $f(\lambda(A^{\hat{u}}(\bar{y}))) = 1$, we have, by (19), that $\beta \leqslant C\delta_1^{(2-n)/4}$, where C depends only on (M,g). Again by (23), we have $\max_M u = \tilde{u}(\tilde{x}) \leqslant C\hat{u}(0) \leqslant C\hat{u}(\bar{y}) = C\xi(\bar{y}) \leqslant C\beta \leqslant C\delta_1^{(2-n)/4}$. Let $\bar{x} \in M$ be a maximum point of u, it was shown in [8] that $f(u(\bar{x})^{-4/(n-2)}\lambda(A_g(\bar{x}))) \leqslant 1$. This, together with (20), implies $\max_M u = u(\bar{x}) \geqslant \delta_2^{(n-2)/4}$. Using the upper bound of $|\nabla_g \log u|$ on M, we have, for some positive constant C depending only on (M,g), that $u \geqslant C^{-1}\max_M u \geqslant C^{-1}\delta_2^{(n-2)/4}$ on M. Step 1 is established.

Step 2. For some constant C depending only on (M,g), b, δ_1 and δ_2, $|\nabla_g^2 u| \leqslant C$ on M.

C^2 estimates for $(f,\Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ were obtained by Viaclovsky [16]. The arguments can be adapted in our situation. Indeed, this is equivalent to setting $\rho \equiv 1$ in the definition of $G(x)$ in the proof of Theorem 1.6 in [8], so that $G(x)$ is defined on M, and Step 2 follows from the computation there (with $h \equiv 1$) together with Step 1. Since f is concave in Γ, and since we have established C^0, C^1 and C^2 estimates of u and u^{-1}, estimate (10) follows from the interior estimates of Evans and Krylov together with the Schauder estimates.

For the existence part of Theorem 1, we only need to treat the case that (M,g) is not conformally diffeomorphic to a standard sphere. The following homotopy was introduced in [8]: for $0 \leqslant t \leqslant 1$, let $f_t(\lambda) = f(t\lambda + (1-t)\sigma_1(\lambda)e)$ be defined on $\Gamma_t := \{\lambda \in \mathbb{R}^n \mid t\lambda + (1-t)\sigma_1(\lambda)e \in \Gamma\}$. We consider, for $0 \leqslant t \leqslant 1$, and for $\hat{g} = u^{4/(n-2)}g$,

$$f_t\big(\lambda(A_{\hat{g}})\big) = 1, \quad \lambda(A_{\hat{g}}) \in \Gamma_t \quad \text{on } M. \tag{24}$$

For $0 \leqslant t \leqslant 1$, (f_t, Γ_t) satisfies (2)–(6) and (7). Moreover estimate (10) holds for solutions of (24), uniform in t. With this the argument in [8] (using the degree theory for second order fully nonlinear elliptic operators in [10]) yields a solution u of (9) in $C^{4,\alpha}$. By standard elliptic theories, $u \in C^\infty(M)$. Theorem 1 is established.

Next we present our proof of Theorem 2. By scaling, it is easy to see that we only need to prove the theorem for $R = \delta = 1$, which we assume below. Let $u(\bar{x}) = \max_{\bar{B}_1} u$. As in the proof of Theorem 1.8 in [8],we can find $\tilde{x} \in B_{1/4}(\bar{x})$ such that

$$u(\tilde{x}) \geqslant 2^{(2-n)/2} \sup_{B_\sigma(\tilde{x})} u \quad \text{and} \quad \gamma := u(\tilde{x})^{2/(n-2)}\sigma \geqslant \frac{1}{2}u(\bar{x})^{2/(n-2)}, \tag{25}$$

where $\sigma = (1-|\tilde{x}-\bar{x}|)/2 \leqslant 1/2$. If $\gamma \leqslant 2^{n+8}n^4$, then $(\sup_{B_1} u)(\inf_{B_2} u) \leqslant u(\bar{x})^2 \leqslant (2\gamma)^{(n-2)/2} \leqslant C(n)$, done. So we assume $\gamma > 2^{n+8}n^4$. Let $\Gamma := u(\tilde{x})^{2/(n-2)} \geqslant 2\gamma$, and consider $w(y) := u(\tilde{x})^{-1}u(\tilde{x} + u(\tilde{x})^{2/(2-n)}y)$, $|y| < \Gamma$. By superharmonicity of u,

$$\min_{\partial B_\Gamma} w = \inf_{B_\Gamma} w \geqslant u(\tilde{x})^{-1}\min_{\partial B_2} u, \qquad 1 = w(0) \geqslant 2^{(2-n)/2}\sup_{B_\gamma} w. \tag{26}$$

We know $F(A^w) = 1$ on B_Γ. Fix $r = 2^{n+6}n^4 < \frac{1}{4}\gamma$. For $|x| < r$, consider $w_{x,\lambda}(y) := (\frac{\lambda}{|y-x|})^{n-2}w(x + \frac{\lambda^2(y-x)}{|y-x|^2})$, $y \in B_\Gamma$. By the conformal invariance of the equation, we have $F(A^{w_{x,\lambda}}) = 1$ on $B_\Gamma \setminus B_\lambda(x)$. As in [8], there exists $0 < \lambda_x < r$ such that $w_{x,\lambda}(y) \leqslant w(y)$, $\forall 0 < \lambda < \lambda_x$, $y \in B_\Gamma \setminus B_\lambda(x)$, and $w_{x,\lambda}(y) < w(y)$, $\forall 0 < \lambda < \lambda_x$, $y \in \partial B_\Gamma$. By the moving sphere argument in [8], we only need to treat two cases:

Case 1. For some $|x| < r$ and some $\lambda \in (0,r)$, $w_{x,\lambda}$ touches w on ∂B_Γ.

Case 2. For all $|x| < r$, all $\lambda \in (0,r)$, we have $w_{x,\lambda}(y) \leqslant w(y)$, $\forall |y-x| \geqslant \lambda$, $y \in B_\Gamma$.

In Case 1, let $\lambda \in (0, r)$ be the smallest number for which $w_{x,\lambda}$ touches w on ∂B_Γ. By (26), we have, for some $|y_0| = \Gamma$, $u(\tilde{x})^{-1} \min_{\partial B_2} u \leqslant \min_{\partial B_\Gamma} w = w_{x,\lambda}(y_0)$. Using (26), $w_{x,\lambda}(y_0) \leqslant 2^{(n-2)/2}(r/(\Gamma - r))^{n-2}$. Therefore,

$$\sigma^{(n-2)/2} u(\tilde{x}) \min_{\partial B_2} u \leqslant 2^{(n-2)/2} \sigma^{(n-2)/2} u(\tilde{x})^2 \frac{r^{n-2}}{(\Gamma/2)^{n-2}} = 2^{(3/2)(n-2)} \sigma^{(n-2)/2} r^{n-2} \leqslant C(n). \tag{27}$$

We deduce from (25) and (27) that $(\sup_{B_1} u)(\inf_{B_2} u) \leqslant 8^{n-2} r^{n-2} \leqslant C(n)$.

In Case 2, we have, by Lemma 1 and (26), that $|\nabla w(y)| \leqslant 2(n-2) r^{-1} w(y) \leqslant (n-2) 2^{n/2} r^{-1}$, $\forall |y| \leqslant r$. Let ε be the number such that $\xi(y) := \frac{1-\varepsilon}{r}(r - |y|^2)$ satisfies $w \geqslant \xi$ on $B_{\sqrt{r}}$ and for some $|\bar{y}| < \sqrt{r}$, $w(\bar{y}) = \xi(\bar{y})$. Since $1 = w(0) \geqslant \xi(0) = 1 - \varepsilon$ and $\xi(\bar{y}) > 0$, we have $0 \leqslant \varepsilon < 1$. By the estimates of $|\nabla w|$ and the mean value theorem, $|w(y) - 1| \leqslant (n-2) 2^{n/2} r^{-1/2}$, for all $|y| \leqslant \sqrt{r}$. So $\frac{1}{2} \leqslant 1 - (n-2) 2^{n/2} r^{-1/2} \leqslant w(\bar{y}) = \xi(\bar{y}) \leqslant 1 - \varepsilon$, and therefore $0 \leqslant \varepsilon \leqslant (n-2) 2^{n/2} r^{-1/2}$. Clearly, $\nabla w(\bar{y}) = \nabla \xi(\bar{y})$, $|\nabla \xi(\bar{y})| \leqslant \frac{2}{\sqrt{r}}$, $D^2 w(\bar{y}) \geqslant D^2 \xi(\bar{y}) = -2(1-\varepsilon) r^{-1} I$. It follows that $A^w(\bar{y}) \leqslant A^\xi(\bar{y}) \leqslant \frac{(10n+4)}{(n-2)^2} 2^{2n/(n-2)} r^{-1} I$. Since $F(A^w(\bar{y})) = 1$, we have, by (16), $\frac{(10n+4)}{(n-2)^2} 2^{2n/(n-2)} r^{-1} \geqslant 1$, violating the choice of r. Thus we have shown that Case 2 can never occur. Theorem 2 is established.

The results in this Note have been presented by the second author at his 45-minute invited talk at ICM 2002 in August 2002 in Beijing. The results have also been presented by the second author in a colloquium talk at Northwestern University on September 27, 2002, in the Geometric Analysis seminar at Princeton University on October 18, 2002, in a mini-course in late October 2002 at Università di Milano. On December 2 2002, the second author was informed by P. Guan that he, in collaboration with C.S. Lin and G. Wang, has obtained some related results.

Note added in proof

We have recently established general Liouville type theorems for conformally invariant fully nonlinear equations (arXiv: math.AP/0301239, arXiv: math.AP/0301254, [9]).

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