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Partial Differential Equations

Limit at infinity for travelling waves in the Gross–Pitaevskii equation

Limite à l’infini des ondes progressives dans l’équation de Gross–Pitaevskii

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Abstract

We study the decay of the travelling waves of finite energy in the Gross–Pitaevskii equation in dimension greater than three and prove their uniform convergence to a constant of modulus one at infinity. **To cite this article:** *P. Gravejat, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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Résumé

Nous étudions la limite à l’infini des ondes progressives d’énergie finie dans l’équation de Gross–Pitaevskii en dimension supérieure ou égale à trois et nous montrons leur convergence uniforme vers une constante de module un. **Pour citer cet article :** *P. Gravejat, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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Version française abrégée

Dans cet article, nous étudions les ondes progressives u de vitesse $c > 0$ pour l’équation de Gross–Pitaevskii $i\partial_t u = \Delta u + u(1 - |u|^2)$ de la forme $u(t, x) = v(x_1 - ct, \dots, x_N)$. L’équation vérifiée par v que nous étudierons désormais est

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0.$$

L’équation de Gross–Pitaevskii est un modèle physique qui décrit la superconductivité et la superfluidité et qui est associé à l’énergie : $E(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (1 - |v|^2)^2$.

Jones et Roberts [7] se sont intéressés aux ondes progressives d’énergie finie parce qu’elles sont supposées expliquer la dynamique en temps long des solutions générales : ils ont ainsi conjecturé qu’elles n’existent que

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lorsque $0 < c < \sqrt{2}$, ce que nous supposons désormais, et qu'elles ont une limite à l'infini qui est une constante de module un.

Béthuel et Saut [2,3] les ont étudiées sur le plan mathématique et ont notamment montré leur existence en dimension deux lorsque c est petit, et l'existence de leur limite à l'infini :

Théorème 0.1. *En dimension deux, une onde progressive pour l'équation de Gross–Pitaevskii de vitesse $0 < c < \sqrt{2}$ et d'énergie finie vérifie à une constante multiplicative de module un près $v(x) \xrightarrow{|x| \rightarrow +\infty} 1$.*

En dimension trois, Béthuel, Orlandi et Smets [1] ont prouvé leur existence lorsque c est petit, et, en toute dimension, Farina [5] a donné une borne universelle sur leur module. Dans cet article, nous allons compléter leurs travaux en dimension supérieure ou égale à trois par le théorème suivant :

Théorème 0.2. *En dimension supérieure ou égale à trois, une onde progressive pour l'équation de Gross–Pitaevskii de vitesse $0 < c < \sqrt{2}$ et d'énergie finie vérifie à une constante multiplicative de module un près $v(x) \xrightarrow{|x| \rightarrow +\infty} 1$.*

Dans la suite, nous esquisserons la preuve de ce théorème : nous déterminerons d'abord la régularité des ondes progressives avant d'énoncer un argument général pour l'étude de la limite à l'infini d'une fonction.

1. Introduction

In this article, we will focus on the travelling waves of speed $c > 0$ in the Gross–Pitaevskii equation $i\partial_t u = \Delta u + u(1 - |u|^2)$ which are of the form $u(t, x) = v(x_1 - ct, \dots, x_N)$. The simplified equation for v , which we will study now, is

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \quad (1)$$

The Gross–Pitaevskii equation is a physical model for superconductivity and superfluidity which is associated to the energy: $E(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (1 - |v|^2)^2$.

The travelling waves of finite energy are supposed to explain the long time dynamics of general solutions and were first considered by Jones and Roberts [7]: they conjectured that they only exist when $c < \sqrt{2}$, which will be supposed henceforth, and that they have a limit at infinity which is a constant of modulus one.

Béthuel and Saut [2,3] first studied mathematically these travelling waves: they showed their existence in dimension two when c is small, and also gave a mathematical proof for their decay at infinity. In fact, they proved the following theorem:

Theorem 1.1. *In dimension two, a travelling wave for the Gross–Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \xrightarrow{|x| \rightarrow +\infty} 1.$$

In dimension three, Béthuel, Orlandi and Smets [1] showed their existence when c is small, and in every dimension, Farina [5] proved a universal bound for their modulus. In this paper, we will complete these results for the dimensions greater than three by proving the following theorem

Theorem 1.2. *In dimension greater than three, a travelling wave for the Gross–Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \xrightarrow{|x| \rightarrow +\infty} 1.$$

This paper will be organized around the proof of this theorem: in a first part, we will study the local and Sobolev regularity of the travelling waves and, in a second part, we will give a general argument to study their decay at infinity.

2. Regularity of travelling waves

In this part, we will study the regularity of a travelling wave of finite energy and of speed $0 < c < \sqrt{2}$ in dimension N greater than two: we will prove the following proposition thanks to arguments from Béthuel and Saut [2,3].

Proposition 2.1. *If v is a solution of Eq. (1) in $L^1_{\text{loc}}(\mathbb{R}^N)$ of finite energy, then v is regular, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in]1, +\infty[$.*

Proof. We begin by establishing the following lemma which is valid even if $c \geq \sqrt{2}$:

Lemma 2.2. *v is regular, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [2, +\infty[$.*

The proof of this lemma is adapted from a bootstrap argument introduced in the article of Béthuel and Saut [2], so, we will only give its sketch, and only in dimension three because the general proof is identical with small changes of Sobolev indices.

We first consider a point z_0 in \mathbb{R}^3 and we denote Ω , the unit ball with center z_0 . Then, we consider the solutions v_1 and v_2 of the equations

$$\begin{cases} \Delta v_1 = 0 & \text{on } \Omega, \\ v_1 = v & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v_2 = g(v) := v(1 - |v|^2) + ic\partial_1 v & \text{on } \Omega, \\ v_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the energy of v is finite, $g(v)$ is uniformly bounded in $L^{4/3}(\Omega)$, which means that $\|g(v)\|_{L^{4/3}(\Omega)}$ is bounded by a constant which only depends on c and $E(v)$ but not on z_0 . By standard elliptic theory, and Sobolev embeddings, v_1 and v_2 are also uniformly bounded in $L^4(\Omega)$ and $W^{2,4/3}(\Omega)$ respectively.

If we denote ω , the ball with center z_0 and with radius $\frac{1}{2}$, by Caccioppoli inequalities, v_1 is uniformly bounded in $W^{2,4/3}(\omega)$ and in $W^{3,12/11}(\omega)$, so, v is uniformly bounded in $W^{2,4/3}(\omega)$. Furthermore, we compute $\nabla g(v) = \nabla v(1 - |v|^2) - 2(v \cdot \nabla v)v + ic\partial_1 \nabla v$, and then, $\nabla g(v)$ is uniformly bounded in $L^{12/11}(\omega)$. By standard elliptic theory, and Sobolev embeddings, we finally get that v is uniformly bounded in $C^{0,1/12}(\omega)$.

Thus, v is continuous and bounded on \mathbb{R}^3 . Then, its gradient $w = \nabla v$ satisfies

$$-\Delta w - ic\partial_1 w + \left(\frac{c^2}{2} + 2\right)w = w(1 - |v|^2) - 2(v \cdot w)v + \left(\frac{c^2}{2} + 2\right)w = h(w),$$

so, from the previous inequalities, $h(w)$ belongs to $L^2(\mathbb{R}^3)$, which proves that w belongs to $H^2(\mathbb{R}^3)$. So w is continuous and bounded, and by iterating, we can conclude that v is regular, bounded and that all its derivatives belong to the spaces $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$. Then, we end this proof by using a standard interpolation result between L^p -spaces.

We deduce from this first lemma the following lemma:

Lemma 2.3. *The modulus ρ of v satisfies*

$$\rho(x) \xrightarrow{|x| \rightarrow +\infty} 1.$$

Indeed, if we denote $\eta = 1 - \rho^2$, η^2 is uniformly continuous because v is bounded and Lipschitzian by Lemma 2.2. As $\int_{\mathbb{R}^N} \eta^2$ is finite, η converges uniformly to 0 at infinity which ends the proof of this lemma.

Thus, ρ does not vanish at the neighborhood of infinity: so, we can write there $v = \rho e^{i\theta}$ and compute the following equations satisfied by ρ and θ :

$$\begin{cases} \operatorname{div}(\rho^2 \nabla \theta) = -\frac{c}{2} \partial_1 \rho^2, \\ -\Delta \rho + \rho |\nabla \theta|^2 + c \rho \partial_1 \theta = \rho(1 - \rho^2). \end{cases} \tag{2}$$

Thanks to this polar form, we can now conclude the proof of Proposition 2.1 by the following lemma:

Lemma 2.4. *The gradient of v belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in]1, 2[$.*

This proof is also adapted from an article of Béthuel and Saut [3], and so, we will only give its sketch. We first notice by Lemma 2.3 that ρ does not vanish at the neighborhood of infinity, and, in order to simplify, we will suppose that ρ does not vanish on \mathbb{R}^N : the general situation is technically slightly more involved, but follows essentially the same idea (see [6]).

So, we begin by denoting $\alpha = 2\eta^2 - 2c\eta\partial_1\theta + 2|\nabla v|^2$ and $R = \eta\nabla\theta$. Because $|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2$, and by Lemmas 2.2 and 2.3, we can establish that α and R are in all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [1, +\infty[$. Besides, we compute thanks to (2.1) using the Fourier transformation

$$\forall \xi \in \mathbb{R}^N, \quad \begin{cases} (|\xi|^2 + 2)\hat{\eta}(\xi) - 2ic\xi_1\hat{\theta}(\xi) = \hat{\alpha}(\xi), \\ |\xi|^2\hat{\theta}(\xi) + \frac{ic}{2}\xi_1\hat{\eta}(\xi) = -i\sum_{j=1}^N \xi_j \widehat{R}_j(\xi). \end{cases}$$

Denoting L and $(L_{j,1})_{1 \leq j \leq N}$ the operators associated to the Fourier multipliers $K(\xi) = |\xi|^2 / (|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)$, respectively $K_{j,1}(\xi) = \xi_j \xi_1 / |\xi|^2$, we can assert

$$\eta = L \left(\alpha + 2c \sum_{j=1}^N L_{j,1}(R_j) \right).$$

Furthermore, the Riesz operator theory checks that the operators $(L_{j,1})_{1 \leq j \leq N}$ are multipliers on all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$, and, K is a regular bounded function on $\mathbb{R}^N \setminus \{0\}$ which satisfies

$$\prod_{j=1}^N (\xi_j^{k_j}) \partial_1^{k_1} \dots \partial_N^{k_N} K(\xi) \in L^\infty(\mathbb{R}^N)$$

as soon as $(k_1, \dots, k_N) \in \{0, 1\}^N$ satisfies $0 \leq \sum_{j=1}^N k_j \leq N$. Therefore, by Lizorkin theorem [8] (see also [9]), L is a multiplier on all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$ too. By the previous statements on α and R , we conclude that η is in all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$, and by the equation

$$\forall j \in \{1, \dots, N\}, \quad \partial_j \theta = -\frac{ic}{2} L_{j,1}(\eta) - i \sum_{k=1}^N L_{j,k}(R_k),$$

where $(L_{j,k})_{1 \leq j,k \leq N}$ is the operator associated to the Fourier multiplier $K_{j,k}(\xi) = \xi_j \xi_k / |\xi|^2$, $\nabla\theta$ is also in all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$.

By iterating this process to all the derivatives of η and $\nabla\theta$ by Lemma 2.2, we conclude that η and $\nabla\theta$ belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in]1, +\infty[$. Since $\eta = 1 - \rho^2$ and ρ is in all the spaces $W^{k,\infty}(\mathbb{R}^N)$ for $k \in \mathbb{N}$, and since $|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2$, Lemma 2.4 is proved as well as Proposition 2.1. \square

3. Decay at infinity

Before concluding the proof of Theorem 1.2, we will establish the following general proposition concerning the decay of a function at infinity.

Proposition 3.1. *We consider a regular function v on \mathbb{R}^N : we suppose that N is greater than three and that the gradient of v belongs to the spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$ where $1 < p_0 < N - 1 < p_1 < +\infty$. Then there is a constant $v_\infty \in \mathbb{C}$ which satisfies*

$$v(x) \xrightarrow{|x| \rightarrow +\infty} v_\infty.$$

Proof. We begin by constructing the limit v_∞ . Indeed, we have

$$\int_{\mathbb{S}^{N-1}} \int_1^{+\infty} |\partial_r v(r\xi)| \, dr \, d\sigma \leq \int_{\mathbb{S}^{N-1}} \left(\int_1^{+\infty} |\nabla v(r\xi)|^{p_0} r^{N-1} \, dr \right)^{1/p_0} \left(\int_1^{+\infty} r^{-(N-1)/(p_0-1)} \, dr \right)^{1/p_0'} \, d\sigma < +\infty$$

which gives $\int_1^{+\infty} |\partial_r v(r\xi)| \, dr < +\infty$ a.e. Thus, there is a function v_∞ defined on \mathbb{S}^{N-1} such that

$$v(r\xi) \xrightarrow{r \rightarrow +\infty} v_\infty(\xi) \quad \text{a.e.}$$

If we denote $\forall p \in [p_0, p_1], \forall r \in \mathbb{R}_+^*, \phi_p(r) = r^{N-1} \int_{\mathbb{S}^{N-1}} |\nabla v(r\xi)|^p \, d\sigma$, this function is regular on \mathbb{R}_+^* and its derivative satisfies

$$\int_0^{+\infty} |\phi_p'(r)| \, dr \leq C (\|\nabla v\|_{L^p(\mathbb{R}^N)}^p + \|\nabla v\|_{L^p(\mathbb{R}^N)}^{p-1} \|\nabla v\|_{W^{1,p}(\mathbb{R}^N)}) < +\infty.$$

Hence, the function ϕ_p has a limit at infinity, and since $\int_0^{+\infty} \phi_p(r) \, dr = \|\nabla v\|_{L^p(\mathbb{R}^N)}^p < +\infty$, this limit is zero. Furthermore, if we denote $\forall(r, \xi) \in \mathbb{R}_+^* \times \mathbb{S}^{N-1}, v_r(\xi) = v(r\xi)$, we remark that $|\nabla v(r\xi)|^2 = |\partial_r v(r\xi)|^2 + r^{-2} |\nabla_{\mathbb{S}^{N-1}} v_r(\xi)|^2$, which leads finally to

$$r^{N-1-p} \int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} v_r(\xi)|^p \, d\sigma \xrightarrow{r \rightarrow +\infty} 0.$$

Thus, if $N - 1 \leq q < \min\{p_1, N\}$, we get for every $r \in \mathbb{R}_+^*$

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} |v_r - v_\infty|^q \, d\sigma &\leq \int_{\mathbb{S}^{N-1}} \left(\int_r^{+\infty} |\partial_r v(s\xi)| \, ds \right)^q \, d\sigma \leq \int_{\mathbb{S}^{N-1}} r^{q-n} \int_r^{+\infty} |\nabla v(s\xi)|^q s^{N-1} \, ds \, d\sigma \\ &\leq C_{n,q} \|\nabla v\|_{L^q(\mathbb{R}^N)}^q r^{q-N}, \end{aligned}$$

which gives

$$\begin{aligned} \|v_r - v_\infty\|_{L^{N-1,1}(\mathbb{S}^{N-1})} &= C_N \int_0^{|\mathbb{S}^{N-1}|} t^{-(N-2)/(N-1)} |v_r - v_\infty|^*(t) \, dt \leq C_{N,q} \left(\int_0^{|\mathbb{S}^{N-1}|} |v_r - v_\infty|^{*q}(t) \, dt \right)^{1/q} \\ &\leq C_{N,q} \|v_r - v_\infty\|_{L^q(\mathbb{S}^{N-1})} \leq C_{N,q} \|\nabla v\|_{L^q(\mathbb{R}^N)}^q r^{q-N} \end{aligned}$$

and proves that $\|v_r - v_\infty\|_{L^{N-1,1}(\mathbb{S}^{N-1})}$ tends to 0 when r tends to $+\infty$. Now, we fix $\varepsilon > 0$ and we denote

$$\forall r \in \mathbb{R}_+, \quad \begin{cases} \forall \lambda \in \mathbb{R}_+^*, & a_r(\lambda) = |\{\xi \in \mathbb{S}^{N-1} \mid |\nabla_{\mathbb{S}^{N-1}} v_r(\xi)| > \lambda\}|, \\ \forall t \in \mathbb{R}_+^*, & f_r(t) = |\nabla_{\mathbb{S}^{N-1}} v_r|^*(t) = \inf\{\lambda \in \mathbb{R}_+^* \mid a_r(\lambda) \leq t\}. \end{cases}$$

We have showed that there exists $r_\varepsilon \in \mathbb{R}_+^*$ such that

$$\forall r > r_\varepsilon, \forall i \in \{0, 1\}, \quad r^{N-1-p_i} \int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} v_r(\xi)|^{p_i} d\sigma \leq \varepsilon^{p_i}.$$

This gives

$$\begin{cases} \forall \lambda \in \mathbb{R}_+^*, & a_r(\lambda) \leq \min\left\{\frac{\varepsilon^{p_0}}{r^{N-1-p_0} \lambda^{p_0}}, \frac{\varepsilon^{p_1}}{r^{N-1-p_1} \lambda^{p_1}}\right\}, \\ \forall t \in \mathbb{R}_+^*, & f_r(t) \leq \min\left\{\frac{\varepsilon}{r^{(N-1)/p_0-1} t^{1/p_0}}, \frac{\varepsilon}{r^{(N-1)/p_1-1} t^{1/p_1}}\right\}. \end{cases}$$

Thus, we finally get

$$\begin{aligned} & \|\nabla_{\mathbb{S}^{N-1}} v_r\|_{L^{N-1,1}(\mathbb{S}^{N-1})} \\ & \leq C_N \varepsilon \left(r^{1-(N-1)/p_1} \int_0^{r^{1-N}} t^{-(N-2)/(N-1)-1/p_1} dt + r^{1-(N-1)/p_0} \int_{r^{1-N}}^{|\mathbb{S}^{N-1}|} t^{-(N-2)/(N-1)-1/p_0} dt \right) \leq C_{N,p_0,p_1} \varepsilon. \end{aligned}$$

This proves that $\nabla_{\mathbb{S}^{N-1}} v_r$ converges to 0 in $L^{N-1,1}(\mathbb{S}^{N-1})$ when r tends to $+\infty$. Since $(v_r)_{r>0}$ converges to v_∞ in $L^{N-1,1}(\mathbb{S}^{N-1})$ and the limit of its gradient is 0 in this same space, we conclude that the gradient of v_∞ is 0 and so, v_∞ is constant. Besides, by a theorem of Cianchi and Pick [4], we know that there is a constant C which satisfies for all $r > 0$

$$\|v_r - v_\infty\|_{L^\infty(\mathbb{S}^{N-1})} \leq C \left(\|v_r - v_\infty\|_{L^{N-1,1}(\mathbb{S}^{N-1})} + \|\nabla_{\mathbb{S}^{N-1}}(v_r - v_\infty)\|_{L^{N-1,1}(\mathbb{S}^{N-1})} \right) \xrightarrow{r \rightarrow +\infty} 0$$

which ends the proof of this proposition. \square

Now, we conclude the proof of Theorem 1.2: if v is a travelling wave of finite energy and of speed $c < \sqrt{2}$, it satisfies the hypothesis of Proposition 3.1 by Proposition 2.1. So there is a constant $v_\infty \in \mathbb{C}$ such that $v(x) \xrightarrow{|x| \rightarrow +\infty} v_\infty$. It remains to show that v_∞ has a modulus equal to one which is clear in view of Lemma 2.3.

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