

# Global solution of nematic liquid crystals models \*

Francisco Guillén-González <sup>a</sup>, Marko Rojas-Medar <sup>b</sup>

<sup>a</sup> Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla,  
Aptdo. 1160, 41080 Sevilla, Spain

<sup>b</sup> Departamento de Matematica Aplicada, IMECC-UNICAMP, C.P. 6065, 13081-970, Campinas-SP, Brazil

Received 29 July 2002; accepted 25 October 2002

Note presented by Philippe G. Ciarlet.

---

## Abstract

We prove existence of a global weak solution for a nematic liquid crystal problem by means of a penalization method using a simplified Ericksen–Leslie model and a new compactness property for the gradient of the director field. *To cite this article: F. Guillén-González, M. Rojas-Medar, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1085–1090.*

© 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## Solution globale pour modes de cristaux liquides nématiques

## Résumé

Par une méthode de pénalisation à partir d'un problème simplifiée de type Ericksen–Leslie, nous démontrons l'existence de solution globale pour le modèle limite de cristaux liquides nématiques. Le point essentiel de la démonstration est une nouvelle propriété de compacité du gradient du vecteur directeur. *Pour citer cet article: F. Guillén-González, M. Rojas-Medar, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1085–1090.*

© 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

---

## Version française abrégée

Dans cette Note, on analyse le caractère globalement bien posé du modèle de crystal liquide nématique (formulé par exemple dans [7]) en utilisant une méthode de pénalisation à partir d'un problème simplifié de type Ericksen–Leslie avec une approximation de Ginzburg–Landau [1,2].

Cette version simplifiée de type Ericksen–Leslie a été introduite par Lin dans [4] puis analysée par Lin et Liu [5,6] par une méthode de Galerkin modifiée et par Shkoller [8] par une méthode de contraction en temps petit suivie d'estimations a priori sur des énergies appropriées. Ce problème est un système d'évolution couplant les équations de Navier–Stokes et de Ginzburg–Landau (que représente la cristallinité du liquide). Coutand and Shkoller dans [3] ont étudié un modèle de type Ericksen–Leslie plus complet, en démontrant l'existence d'une (unique) solution en temps petit (globale pour des données initiales suffisamment petites). Dans cette Note, nous étudions le comportement asymptotique par rapport le parameter de penalisation.

Les inconnues sont la vitesse  $\mathbf{u}(t, \mathbf{x})$  et la pression  $p(t, \mathbf{x})$  du fluide et le vecteur directeur  $\mathbf{d}(t, \mathbf{x})$  pour l'orientation du cristal liquide, situé dans un ouvert régulier  $\Omega \subset \mathbb{R}^n$  ( $n = 2$  ou  $3$ ).

---

*E-mail addresses:* guillen@numer.us.es (F. Guillén-González); marko@ime.unicamp.br (M. Rojas-Medar).

Dans le modèle pénalisé, on a la restriction  $|\mathbf{d}| \leq 1$  comme conséquence d'un principe de maximum pour les équations de Ginzburg–Landau avec l'approximation

$$\mathbf{f}_\varepsilon(\mathbf{d}) = \varepsilon^{-2}(|\mathbf{d}|^2 - 1)\mathbf{d},$$

où  $|\mathbf{d}| = |\mathbf{d}(t, \mathbf{x})|$  dénote la norme Euclidienne ponctuelle dans  $\mathbb{R}^n$ . Cette fonction de pénalisation a une structure potentielle, i.e. il existe une fonction potentielle  $F_\varepsilon(\mathbf{d}) = \varepsilon^{-2}(|\mathbf{d}|^2 - 1)^2$  vérifiant  $\mathbf{f}_\varepsilon(\mathbf{d}) = \nabla_{\mathbf{d}}(F_\varepsilon(\mathbf{d}))$  pour tout  $\mathbf{d} \in \mathbb{R}^n$ .

Donc, on considère le modèle pénalisé (1)–(5) dans  $(0, T) \times \Omega$  (nous renvoyons le lecteur à la version anglaise pour le modèle et aussi pour les notations utilisées, les données du problème et les espaces de fonctions associés).

Pour le modèle limite (quand  $\varepsilon$  vers zéro), nous trouverons la restriction  $|\mathbf{d}| = 1$  et le multiplicateur de Lagrange associé  $|\nabla \mathbf{d}|^2 \mathbf{d}$ . Alors, dans le problème limite, on change (1) par (6).

Nous donnons le résultat principal de la Note

**THÉORÈME 0.1.** – Soit  $T > 0$  et  $\Omega \subset \mathbb{R}^n$  un ouvert borné de classe  $C^2$ . Si on impose  $\mathbf{u}_0 \in \mathbf{H}$ ,  $\mathbf{d}_0 \in \mathbf{H}_h^1$  et  $\mathbf{h} \in \mathbf{H}^2$  tel que  $|\mathbf{d}_0| = 1$  dans  $\Omega$  et  $|\mathbf{h}| = 1$  sur  $\partial\Omega$ , alors il existe une solution globale faible  $\mathbf{u} \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{H})$ ,  $\mathbf{d} \in L^\infty(0, T; \mathbf{H}_h^1)$  du problème limite (2)–(6), qui est obtenu comme limite de solutions « semi-fort »  $\mathbf{u}_\varepsilon \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{H})$ ,  $\mathbf{d}_\varepsilon \in L^2(0, T; \mathbf{H}^2 \cap \mathbf{H}_h^1) \cap L^\infty(0, T; \mathbf{H}_h^1)$  du problème pénalisé de Navier–Stokes et Ginzburg–Landau (1)–(5) quand  $\varepsilon$  vers zéro.

*Remarque 1.* – D'après la littérature existante, cette théorème est le première resultat sur l'existence de solution globale en temps (sans restrictions sur les données) du problème limite (2)–(6). Dans [7], Prohl démontre l'existence (et unicité) d'une solution locale en temps de ce problème. D'autre part, Lin et Liu ont étudié dans [6] le comportement asymptotique du problème pénalisé (1)–(5) quand  $\varepsilon$  vers zéro, mais sans contrôler le limite des termes (fortement) non linéaire  $\nabla \mathbf{d}_\varepsilon \odot \nabla \mathbf{d}_\varepsilon$  de (2). Ils prouvent seulement la convergence vers un tenseur mesure-valué  $M$ . Notre principal contribution dans cette Note est l'identification de  $M$  avec le tenseur limite  $\nabla \mathbf{d} \odot \nabla \mathbf{d}$ .

*Remarque 2.* – Remarquer que dans le problème limite on perd la régularité  $H^2$  du vecteur directeur  $\mathbf{d}$ , donc  $(\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon)$  vérifie (1) ponctuellement presque partout  $(t, \mathbf{x})$ , mais le limite  $(\mathbf{u}, \mathbf{d})$  vérifie (6) dans le sens des distributions.

## 1. Introduction

In this Note, we establish the well-posedness in the large of a nematic liquid crystals model (formulated for instance in [7]) by means of a penalisation argument using a simplified Ericksen–Leslie model with the Ginzburg–Landau approximation [1,2].

Let us consider a simplified version of the Ericksen–Leslie model, introduced by Lin in [4] and analysed by Lin and Liu [5,6] who used a modified Galerkin approach, and by Shkoller [8] who relied on a contraction mapping argument coupled with appropriate energy estimates. This model is a modified Navier–Stokes system that take into account of the liquid crystallinity, coupling with the Ginzburg–Landau equations. A full version of this Ericksen–Leslie model has been recently studied by Coutand and Shkoller in [3], where local well-posedness (global for small enough data) is proved. Now, we are interested in the asymptotic behaviour respect to the penalisation parameter.

The unknowns are the time-dependent divergence-free velocity field  $\mathbf{u}(t, \mathbf{x})$  and pressure  $p(t, \mathbf{x})$  of the fluid and the director field  $\mathbf{d}(t, \mathbf{x})$  representing the orientation of the liquid crystals molecules. The fluid is confined in an open bounded domain  $\Omega \subset \mathbb{R}^n$  ( $n = 2$  or  $3$ ) with boundary  $\partial\Omega$  of  $C^2$  type.

In the penalised model one verifies the constraint  $|\mathbf{d}| \leq 1$  as consequence of a maximum principle for the Ginzburg–Landau equation where the approximation

$$\mathbf{f}_\varepsilon(\mathbf{d}) = \varepsilon^{-2}(|\mathbf{d}|^2 - 1)\mathbf{d}$$

is considered ( $\varepsilon > 0$ ). Here  $|\mathbf{d}| = |\mathbf{d}(t, \mathbf{x})|$  denotes the punctual Euclidean norm in  $\mathbb{R}^n$ . This penalisation function exhibits potential structure, i.e., there exists a potential function

$$F_\varepsilon(\mathbf{d}) = \varepsilon^{-2}(|\mathbf{d}|^2 - 1)^2$$

such that  $f_\varepsilon(\mathbf{d}) = \nabla_{\mathbf{d}}(F_\varepsilon(\mathbf{d}))$  for all  $\mathbf{d} \in \mathbb{R}^n$ .

Accordingly, we consider the penalised model in  $(0, T) \times \Omega$  as follows

$$|\mathbf{d}| \leq 1, \quad \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} + \gamma(\mathbf{f}_\varepsilon(\mathbf{d}) - \Delta \mathbf{d}) = 0, \tag{1}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p + \lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) = 0, \tag{2}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{d}|_{\partial\Omega} = \mathbf{h}, \tag{4}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{d}|_{t=0} = \mathbf{d}_0. \tag{5}$$

Here,  $\mathbf{u}_0$  and  $\mathbf{d}_0$  are, respectively, the initial velocity and director fields. In order to obtain the dissipativity of the model, we shall assume (as in all previous works) time independent (Dirichlet) boundary data for the director field  $\mathbf{d}$ , given by  $\mathbf{h} : \partial\Omega \rightarrow \mathbb{R}^n$ . Concerning the coefficients,  $\nu > 0$  represents the viscosity of the fluid,  $\lambda > 0$  is an elasticity constant,  $\varepsilon > 0$  is a penalisation parameter (with respect to the unitary constraint), and  $\gamma > 0$  is a relaxation-time constant. We have used the tensorial notation

$$(\nabla \mathbf{d} \odot \nabla \mathbf{d})_{ij} = \sum_{k=1}^n \partial_{x_i} d_k \partial_{x_j} d_k.$$

In the  $\varepsilon$ -limit model we will find the restriction  $|\mathbf{d}| = 1$  and the Lagrange multiplier associated  $|\nabla \mathbf{d}|^2 \mathbf{d}$ . Indeed, when  $\varepsilon \rightarrow 0$  we will find a limit problem, where it changes (1) by

$$|\mathbf{d}| = 1, \quad \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} - \gamma \Delta \mathbf{d} - \gamma |\nabla \mathbf{d}|^2 \mathbf{d} = 0. \tag{6}$$

Let us introduce the following space of functions

$$\mathbf{H}_{\mathbf{h}}^1 = \{ \mathbf{d} \in H^1(\Omega)^n \mid \mathbf{d} = \mathbf{h} \text{ on } \partial\Omega \},$$

$$\mathbf{H} = \{ \mathbf{u} \in L^2(\Omega)^n \mid \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \},$$

$$\mathbf{V} = \{ \mathbf{u} \in H_0^1(\Omega)^n \mid \nabla \cdot \mathbf{u} = 0 \}.$$

For simplicity, let us denote  $\mathbf{L}^2, \mathbf{H}^1$  instead of  $L^2(\Omega)^n, H^1(\Omega)^n$  etc. Our main result is the following

**THEOREM 1.1.** – *Let  $T > 0$  and  $\Omega \subset \mathbb{R}^n$  be an open, bounded and  $C^2$  domain. Let us assume  $\mathbf{u}_0 \in \mathbf{H}$ ,  $\mathbf{d}_0 \in \mathbf{H}_{\mathbf{h}}^1$  and  $\mathbf{h} \in \mathbf{H}^2$  such that  $|\mathbf{d}_0| = 1$  in  $\Omega$  and  $|\mathbf{h}| = 1$  on  $\partial\Omega$ . Then, there exists a global weak solution  $\mathbf{u} \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{H})$ ,  $\mathbf{d} \in L^\infty(0, T; \mathbf{H}_{\mathbf{h}}^1)$  of the limit problem (2)–(6) obtained as a limit of “semi-strong” solutions  $\mathbf{u}_\varepsilon \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{H})$ ,  $\mathbf{d}_\varepsilon \in L^2(0, T; \mathbf{H}^2 \cap \mathbf{H}_{\mathbf{h}}^1) \cap L^\infty(0, T; \mathbf{H}_{\mathbf{h}}^1)$  of the coupled Navier–Stokes and Ginzburg–Landau model (1)–(5) as  $\varepsilon$  goes to zero.*

*Remark 1.* – Up to our known, this theorem is the first result of existence of a global in time solution (without restrictions on the data) of the limit problem (2)–(6). In [7], Prohl proves existence (and uniqueness) of a local in time strong solution of (2)–(6). On the other hand, Lin and Liu studied in [6] the asymptotic behaviour of (1)–(5) when  $\varepsilon$  goes to zero, but the limit of the (strongly) nonlinear terms  $\nabla \mathbf{d}_\varepsilon \odot \nabla \mathbf{d}_\varepsilon$  is only obtained towards a measure valued tensor  $M$ . The main contribution in this note is to identify  $M$  with the limit tensor  $\nabla \mathbf{d} \odot \nabla \mathbf{d}$ .

*Remark 2.* – Notice that in the  $\varepsilon$ -limit problem (2)–(6) one loses the  $H^2$ -regularity for the director field  $\mathbf{d}$ , hence although  $(\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon)$  verifies (1) point-wise a.e.  $(t, \mathbf{x})$ , their limit  $(\mathbf{u}, \mathbf{d})$  verifies (6) only in a distributional sense.

**2.  $\varepsilon$ -approximate solutions and dissipativity**

For each  $\varepsilon > 0$ , let us consider a “semi-strong” solution  $(\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon)$  of the  $\varepsilon$ -approximate problem (1)–(5), that is

$$\mathbf{u}_\varepsilon \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{H}), \quad \mathbf{d}_\varepsilon \in L^2(0, T; \mathbf{H}^2 \cap \mathbf{H}_h^1) \cap L^\infty(0, T; \mathbf{H}_h^1), \quad (7)$$

verifying the  $\mathbf{u}$ -system (2) in the distributional sense and the  $\mathbf{d}$ -system (1) point-wise a.e. The boundary conditions are verified in the trace sense. Finally, the initial conditions have classical sense; indeed  $\mathbf{u}_\varepsilon$  and  $\mathbf{d}_\varepsilon$  are time-continuous functions, as consequence of the additional regularity

$$\partial_t \mathbf{u}_\varepsilon \in L^p(0, T; \mathbf{V}') \quad \text{and} \quad \partial_t \mathbf{d}_\varepsilon \in L^p(0, T; \mathbf{L}^2) \quad (p = 2 \text{ if } n = 2 \text{ or } p = 4/3 \text{ if } n = 3)$$

that can be obtained applying the previous regularity (7) to Eqs. (1) and (2).

The existence of  $(\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon)$  can be proved [5] by means of three main arguments: a *semi-Galerkin* method (space-discretization of the  $\mathbf{u}$ -system (2), remaining the  $\mathbf{d}$ -system (1) in the continuous sense), a *maximum principle* for the  $\mathbf{d}$ -system in order to obtain the constraint  $|\mathbf{d}_\varepsilon| \leq 1$  and the following *energy inequality*,

$$\frac{d}{dt} \left( \frac{1}{2} \|\mathbf{u}_\varepsilon\|^2 + \frac{\lambda}{2} \|\nabla \mathbf{d}_\varepsilon\|^2 + \lambda \int_\Omega F_\varepsilon(\mathbf{d}_\varepsilon) \, d\mathbf{x} \right) + \nu \|\nabla \mathbf{u}_\varepsilon\|^2 + \lambda \gamma \|\mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon) - \Delta \mathbf{d}_\varepsilon\|^2 \leq 0, \quad (8)$$

obtained taking respectively  $\lambda(\mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon) - \Delta \mathbf{d}_\varepsilon)$  and  $\mathbf{u}_\varepsilon$  as test functions in (1), (2) and using the equality  $\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) = \nabla(|\nabla \mathbf{d}|^2/2) + \nabla \mathbf{d}' \Delta \mathbf{d}$ . In (8),  $\|\cdot\|$  denotes the  $L^2(\Omega)$ -norm.

**3.  $\varepsilon$ -independent estimates**

For simplicity, let us denote  $L^2(\mathbf{H}^1)$  instead of  $L^2(0, T; \mathbf{H}^1)$ , etc. From the maximum principle in (2)

$$|\mathbf{d}_\varepsilon| \leq 1 \quad \text{a.e. } (t, \mathbf{x}).$$

On the other hand, from the energy inequality (8), one has the following ( $\varepsilon$ -independent) estimates:

$$\mathbf{u}_\varepsilon \text{ is bounded in } L^\infty(\mathbf{H}) \cap L^2(\mathbf{V}), \quad (9)$$

$$\mathbf{d}_\varepsilon \text{ is bounded in } L^\infty(\mathbf{H}_h^1), \quad (10)$$

$$\mathbf{w}_\varepsilon := \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon) - \Delta \mathbf{d}_\varepsilon \text{ is bounded in } L^2(\mathbf{L}^2), \quad (11)$$

$$F_\varepsilon(\mathbf{d}_\varepsilon) \text{ is bounded in } L^\infty(L^1). \quad (12)$$

Moreover, applying estimates (9)–(11) in Eqs. (1), (2),

$$\partial_t \mathbf{u}_\varepsilon \text{ is bounded in } L^2(\mathbf{V}') + L^\infty((\mathbf{W}^{1,r} \cap \mathbf{V})'), \quad r > n, \quad (13)$$

$$\partial_t \mathbf{d}_\varepsilon \text{ is bounded in } L^2(\mathbf{L}^2) + L^q(\mathbf{L}^{4/3}), \quad q = 4 \text{ if } n = 2 \text{ or } q = 8/3 \text{ if } n = 3. \quad (14)$$

Finally, (10) implies in particular

$$M_\varepsilon := \nabla \mathbf{d}_\varepsilon \odot \nabla \mathbf{d}_\varepsilon \text{ is bounded in } L^\infty(\mathbf{L}^1). \quad (15)$$

**4.  $\varepsilon$ -convergence**

From previous estimates (9)–(15) and compactness results of Aubin–Lions type [9], there exists subsequences (equally denoted)  $\mathbf{d}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{w}_\varepsilon, M_\varepsilon$  and their respective limit functions  $\mathbf{d}, \mathbf{u}, \mathbf{w}, M$ , such that

$$\begin{aligned} \mathbf{u}_\varepsilon &\rightarrow \mathbf{u} \text{ in } L^\infty(\mathbf{H}) \text{ weak}\star, L^2(\mathbf{V}) \text{ weak}, L^2(\mathbf{H}) \text{ strong,} \\ \mathbf{d}_\varepsilon &\rightarrow \mathbf{d} \text{ in } L^\infty(\mathbf{H}^1) \text{ weak}\star, C(\mathbf{L}^2) \text{ strong,} \\ \partial_t \mathbf{u}_\varepsilon &\rightarrow \partial_t \mathbf{u} \text{ in } L^2(\mathbf{V}') + L^\infty((\mathbf{W}^{1,r} \cap \mathbf{V})') \text{ weak,} \\ \partial_t \mathbf{d}_\varepsilon &\rightarrow \partial_t \mathbf{d} \text{ in } L^2(\mathbf{L}^2) + L^p(\mathbf{L}^{4/3}) \text{ weak,} \\ \mathbf{w}_\varepsilon &\rightarrow \mathbf{w} \text{ in } L^2(\mathbf{L}^2) \text{ weak,} \\ \nabla \mathbf{d}_\varepsilon \odot \nabla \mathbf{d}_\varepsilon &\rightarrow M \text{ in the measure sense.} \end{aligned}$$

Consequently, taking limits as  $\varepsilon \rightarrow 0$  and using De Rham's lemma [10], we arrive at

$$\partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} + \gamma \mathbf{w} = 0, \tag{16}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p + \lambda \nabla \cdot M = 0, \tag{17}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{18}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{d}|_{\partial\Omega} = \mathbf{h}, \tag{19}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{d}|_{t=0} = \mathbf{d}_0. \tag{20}$$

On the other hand, since  $\varepsilon^{-1}(|\mathbf{d}_\varepsilon|^2 - 1)$  is bounded in  $L^\infty(\mathbf{L}^2)$  (using (12)) and  $\mathbf{d}_\varepsilon \rightarrow \mathbf{d}$  point-wise a.e.  $(t, \mathbf{x})$ , one get the unity constraint  $|\mathbf{d}(t, \mathbf{x})| = 1$  a.e.  $(t, \mathbf{x})$ . Therefore, in order to finish the proof of Theorem 1.1, we have to identify  $\mathbf{w}$  with  $-\Delta \mathbf{d} - |\nabla \mathbf{d}|^2 \mathbf{d}$  and  $M$  with  $\nabla \mathbf{d} \odot \nabla \mathbf{d}$ .

### 5. Identification of $\mathbf{w} = -\Delta \mathbf{d} - |\nabla \mathbf{d}|^2 \mathbf{d}$

The argument of this section is based in the known literature on harmonic functions with values in the unit spheric surface (see for instance [1,2] and references therein cited). In particular, we will use the following result, which is a slightly modification (introducing the convection terms) of Lemma 2.2 in [2] (see also Lemma 7.1 in [6]):

LEMMA 5.1. – *The following two systems are equivalent:*

$$\partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} - \gamma \Delta \mathbf{d} = \gamma |\nabla \mathbf{d}|^2 \mathbf{d}$$

and

$$|\mathbf{d}| = 1, \quad (\partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d}) \wedge \mathbf{d} - \gamma \nabla \cdot (\nabla \mathbf{d} \wedge \mathbf{d}) = 0. \tag{21}$$

Since we already have that  $|\mathbf{d}| = 1$ , it suffices to verify Eq. (21). Indeed, making the vectorial product of Eq. (2) by  $\mathbf{d}_\varepsilon$ , taking into account that  $\mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon) \wedge \mathbf{d}_\varepsilon = 0$  and  $-\Delta \mathbf{d}_\varepsilon \wedge \mathbf{d}_\varepsilon = -\nabla \cdot (\nabla \mathbf{d}_\varepsilon \wedge \mathbf{d}_\varepsilon)$ , one has

$$(\partial_t \mathbf{d}_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{d}_\varepsilon) \wedge \mathbf{d}_\varepsilon - \gamma \nabla \cdot (\nabla \mathbf{d}_\varepsilon \wedge \mathbf{d}_\varepsilon) = 0. \tag{22}$$

Making  $\varepsilon \rightarrow 0$ , we can deduce (21), using the strong convergences of  $\mathbf{u}_\varepsilon$  and  $\mathbf{d}_\varepsilon$  and the weak convergences of  $\partial_t \mathbf{d}_\varepsilon$  and  $\nabla \mathbf{d}_\varepsilon$ .

### 6. Identification of $M = \nabla \mathbf{d} \odot \nabla \mathbf{d}$

The key of this proof is to obtain  $L^2$ -compactness for  $\nabla \mathbf{d}_\varepsilon$ , using some ideas of the optimisation framework. Indeed, since  $\mathbf{w}_\varepsilon = -\Delta \mathbf{d}_\varepsilon + \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)$ , then  $\mathbf{d}_\varepsilon(t)$  can be viewed as a solution of the optimisation (without constraints) problem

$$J_\varepsilon(\mathbf{d}_\varepsilon(t)) = \min_{\mathbf{d} \in \mathbf{H}_h^1(\Omega)} J_\varepsilon(\mathbf{d}) \left[ = \int_\Omega \left( \frac{1}{2} |\nabla \mathbf{d}|^2 + F_\varepsilon(\mathbf{d}) - \mathbf{w}_\varepsilon(t) \cdot \mathbf{d} \right) \right].$$

On the other hand, a.e.  $t$  let us define  $\tilde{\mathbf{d}}(t)$  as a solution of the optimisation (with constraints) problem

$$J(\tilde{\mathbf{d}}(t)) = \min_{\mathbf{d} \in \mathbf{H}_h^1(\Omega) \mid |\mathbf{d}|=1} J(\mathbf{d}) \left[ = \int_{\Omega} \left( \frac{1}{2} |\nabla \mathbf{d}|^2 - \mathbf{w}(t) \cdot \mathbf{d} \right) \right].$$

Obviously,  $J_{\varepsilon}(\mathbf{d}_{\varepsilon}(t)) \leq J_{\varepsilon}(\tilde{\mathbf{d}}(t))$ . In particular,

$$\int_0^T \int_{\Omega} \left( \frac{1}{2} |\nabla \mathbf{d}_{\varepsilon}(t)|^2 + F_{\varepsilon}(\mathbf{d}_{\varepsilon}(t)) - \mathbf{w}_{\varepsilon}(t) \cdot \mathbf{d}_{\varepsilon}(t) \right) \leq \int_0^T \int_{\Omega} \left( \frac{1}{2} |\nabla \tilde{\mathbf{d}}(t)|^2 - \mathbf{w}_{\varepsilon}(t) \cdot \tilde{\mathbf{d}}(t) \right). \quad (23)$$

Taking limit inf as  $\varepsilon \rightarrow 0$  in (23), bounding previously  $F_{\varepsilon}(\mathbf{d}_{\varepsilon}(t)) \geq 0$ ,

$$\begin{aligned} \int_0^T J(\mathbf{d}(t)) &\leq \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \left( \frac{1}{2} |\nabla \mathbf{d}_{\varepsilon}(t)|^2 - \mathbf{w}_{\varepsilon}(t) \cdot \mathbf{d}_{\varepsilon}(t) \right) \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \left( \frac{1}{2} |\nabla \tilde{\mathbf{d}}(t)|^2 - \mathbf{w}_{\varepsilon}(t) \cdot \tilde{\mathbf{d}}(t) \right) = \int_0^T J(\tilde{\mathbf{d}}(t)), \end{aligned}$$

hence one has the equality  $\int_0^T J(\mathbf{d}(t)) = \int_0^T J(\tilde{\mathbf{d}}(t))$  (the opposite inequality is easy to deduce since  $|\mathbf{d}(t)| = 1$ ). Consequently, all the previous inequalities are equalities, and in particular

$$\exists \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \left( \frac{1}{2} |\nabla \mathbf{d}_{\varepsilon}(t)|^2 - \mathbf{w}_{\varepsilon}(t) \cdot \mathbf{d}_{\varepsilon}(t) \right) = \int_0^T J(\mathbf{d}(t)) = \int_0^T \int_{\Omega} \left( \frac{1}{2} |\nabla \mathbf{d}(t)|^2 - \mathbf{w}(t) \cdot \mathbf{d}(t) \right).$$

Since  $\exists \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \mathbf{w}_{\varepsilon}(t) \cdot \mathbf{d}_{\varepsilon}(t) = \int_0^T \int_{\Omega} \mathbf{w}(t) \cdot \mathbf{d}(t)$  (using the strong convergence of  $\mathbf{d}_{\varepsilon}$ ), one has

$\exists \lim_{\varepsilon \rightarrow 0} \|\nabla \mathbf{d}_{\varepsilon}\|_{L^2(L^2)}^2 = \|\nabla \mathbf{d}\|_{L^2(L^2)}^2$  therefore,

$$\mathbf{d}_{\varepsilon} \rightarrow \mathbf{d} \quad \text{in } L^2(\mathbf{H}^1)\text{-strong.}$$

In particular, taking into account the estimates of  $\nabla \mathbf{d}_{\varepsilon}$ ,

$$\nabla \mathbf{d}_{\varepsilon} \rightarrow \nabla \mathbf{d} \quad \text{in } L^p(L^q)\text{-strong, } \forall p, q : p < \infty, q < 2,$$

hence we can identify  $M = \nabla \mathbf{d} \odot \nabla \mathbf{d}$ .

\* The first author has been partially financed by the projet BFM2000-1317, and the second author by the projects CNPq-Brasil 300116-93-4 and Fapesp-Brasil 01/07557-3.

### References

[1] F. Bethuel, H. Brezis, F. Hélein, Asymptotics for the minimization of a Ginzburg–Landau functional, *Calc. Var.* 1 (1993) 123–148.  
 [2] Y. Chen, The weak solutions to the evolution problems of harmonic maps, *Math. Z.* 201 (1989) 69–74.  
 [3] D. Coutand, S. Shkoller, Well-posedness of the full Ericksen–Leslie model of nematic liquid crystals, *C. R. Acad. Sci. Paris, Série I* 333 (2001) 919–924.  
 [4] F.H. Lin, Nonlinear theory of defects in nematic liquid crystals: phase transition and flow phenomena, *Comm. Pure Appl. Math.* 42 (1989) 789–814.  
 [5] F.H. Lin, C. Liu, Non-parabolic dissipative systems modelling the flow of liquid crystals, *Comm. Pure Appl. Math.* 48 (1995) 501–537.  
 [6] F.H. Lin, C. Liu, Existence of solutions for the Ericksen–Leslie system, *Arch. Rational Mech. Anal.* 154 (2000) 135–156.  
 [7] A. Prohl, *Computational Micro-Magnetism*, in: *Adv. Numer. Math.*, Teubner, 2001.  
 [8] S. Shkoller, Well-posedness and global attractors for liquid crystals on Riemannian manifolds, *Comm. Partial Differential Equations* 27 (5 & 6) (2001) 1103–1137.  
 [9] J. Simon, Compact sets in  $L^p(0, T; B)$ , *Ann. Mat. Pura Appl.* 146 (1987) 65–97.  
 [10] R. Temam, *Navier–Stokes Equations*, North-Holland, Elsevier, 1985.