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Mapping class group of a non-orientable surface and moduli space of Klein surfaces

Błażej Szepietowski [1]

Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland

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Abstract As for Riemann surfaces, the moduli space of closed non-orientable Klein surfaces of genus g can be defined as the orbit space of the Teichmüller space $\mathcal{T}_g$ by the mapping class group Mod_g of a closed non-orientable surface. Using the set of generators given by Birman and Chillingworth, we prove that the latter group is generated by involutions. We conclude, using the Armstrong's result, that the moduli space is simply-connected. ***To cite this article: B. Szepietowski, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1053–1056.***

Groupe modulaire d'une surface non-orientable et l'espace des modules des surfaces de Klein

Résumé Comme pour les surfaces de Riemann, l'espace des modules des surfaces de Klein fermées, non-orientable et de genre g peut être défini comme l'espace des orbites de l'espace de Teichmüller $\mathcal{T}_g$ sous l'action du groupe modulaire Mod_g d'une surface fermée, non-orientable. Utilisant l'ensemble de générateurs donné par Birman et Chillingworth nous prouvons que le dernier groupe est engendré par des involutions. On en déduit, utilisant le résultat d'Armstrong, que l'espace des modules est simplement-connexe. ***Pour citer cet article : B. Szepietowski, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1053–1056.***

1. It is well known that non-orientable surfaces do not admit analytic structures. However, if we also allow anti-analytic functions (i.e., functions which composed with the complex conjugation are analytic) as transition functions, we obtain the concepts of dianalytic structures and Klein surfaces as those surfaces which carry such structures. These concepts were already known to Klein himself but their revitalization is due to Alling and Greenleaf [1].

Given a closed non-orientable Klein surface X of topological genus g there is a unique Riemann double cover $\widetilde{X}$ of genus $g-1$ and an anti-analytic involution σ, such that the canonical dianalytic structure on the orbit space $\widetilde{X}/\sigma$ makes it isomorphic to X [1]. As a result we have a mapping

$$\mathcal{M}_g \to \mathcal{M}^+_{g-1} \tag{1}$$

E-mail address: blaszep@math.univ.gda.pl (B. Szepietowski).

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between the moduli spaces of Klein and Riemann surfaces. Maclachlan showed in [6] that $\mathcal{M}_g^+$ is simply connected and we prove the same for $\mathcal{M}_g$. We do not use the mapping (1) in our proof because it's nature is difficult to understand. For example, it is known that non-isomorphic Riemann surfaces can stand as double covers for different numbers of non-isomorphic Klein surfaces.

2. Let $\mathcal{H}$ denote the upper half plane and Ω the group of all automorphisms of $\mathcal{H}$. An *NEC group* is a discrete, cocompact subgroup of Ω. A non-orientable closed Klein surface X of genus $g \geqslant 3$ can be represented as an orbit space $\mathcal{H}/\Gamma$, where Γ is an NEC group isomorphic to the fundamental group of X and so it has the presentation

$$\left\langle d_1, \ldots, d_g \,\middle|\, \prod_{i=1}^{g} d_i^2 = 1 \right\rangle.$$

Let $\mathcal{R}(\Gamma)$ denote the set of monomorphisms $\alpha : \Gamma \to \Omega$ whose image is an NEC group. The set $\mathcal{R}(\Gamma)$ can be topologized as a subspace of Ω^g. Now Ω acts on $\mathcal{R}(\Gamma)$ by $t(\alpha)(\gamma) = t \circ \alpha(\gamma) \circ t^{-1}$ for $t \in \Omega$ and all $\gamma \in \Gamma$. The orbit space is the Teichmüller space $\mathcal{T}_g$ and is homeomorphic to an open ball in some Euclidean space [5]. The mapping class group Mod_g is isomorphic to the group $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ of the outer automorphisms of Γ and we have a natural action of Mod_g on $\mathcal{T}_g$. The orbit space $\mathcal{M}_g$ is the moduli space which parametrizes the isomorphism classes of closed non-orientable Klein surfaces of genus g. Observe that the mapping (1) assigns to a class of α the class of its restriction to the canonical Fuchsian subgroup Γ^+ of Γ.

3. We now shortly present the main results of Birman and Chillingworth [3]. We will use them in proving Theorem 3 below.

Let $\widetilde{X}$ be an orientable surface of genus $g \geqslant 2$ represented in $\mathbb{R}^3$ in such a way that it is invariant under reflexions about the xy, xz and yz planes, as illustrated in Figs. 1 and 2.

We define a homeomorphism $J : \widetilde{X} \to \widetilde{X}$ by $J(x, y, z) = (-x, -y, -z)$. The quotient space $X = \widetilde{X}/_\sim$ with respect to the relation $(x, y, z) \sim J(x, y, z)$ is a non-orientable surface of topological genus $g + 1$. The projection $p : \widetilde{X} \to X$ is a covering map of degree 2. A homeomorphism $\tilde{h} : \widetilde{X} \to \widetilde{X}$ will be called *symmetric* if $J\tilde{h} = \tilde{h}J$. If $\tilde{h}$ is symmetric, it projects to $h : X \to X$ defined unambiguously by $hp = p\tilde{h}$.

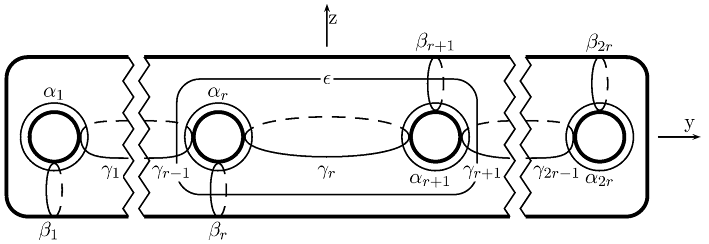

Figure 1. – $g = 2r$.

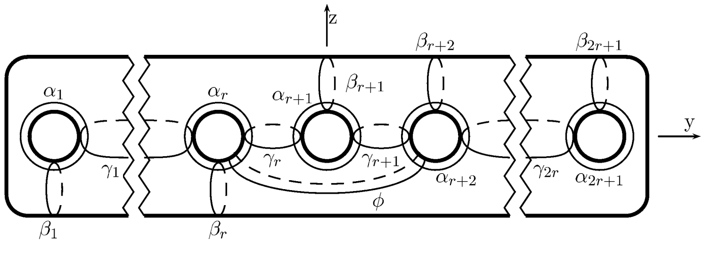

Figure 2. – $g = 2r + 1$.

Let $H(\widetilde{X})$ denote the group of all homeomorphisms $\widetilde{X} \to \widetilde{X}$, $I(\widetilde{X})$ the normal subgroup of homeomorphisms isotopic to the identity and $M(\widetilde{X})$ the factor group $H(\widetilde{X})/I(\widetilde{X})$, i.e., the homeotopy group or the extended mapping class group of $\widetilde{X}$. Let $S(\widetilde{X}) < H(\widetilde{X})$ be the subgroup consisting of all homeomorphisms isotopic to symmetric maps; let $T(\widetilde{X}) < H(\widetilde{X})$ be the subgroup of homeomorphisms isotopic to the covering transformation J or to the identity map.

THEOREM 1 ([3]). – *The projection of symmetric homeomorphisms of $\widetilde{X}$ to homeomorphisms of X induces a map*

$$\big(S(\widetilde{X})/I(\widetilde{X})\big)/\big(T(\widetilde{X})/I(\widetilde{X})\big) \to \mathrm{Mod}_{g+1}$$

which is an isomorphism onto.

Let a_i, b_i, c_k, e, f denote the isotopy classes of twists about the circles $\alpha_i, \beta_j, \gamma_k, \epsilon, \phi$ in Figs. 1 and 2. Let j denote the isotopy class of J.

THEOREM 2 ([3]). – *The subgroup $S(\widetilde{X})/I(\widetilde{X}) < M(\widetilde{X})$ is generated by*:

$$\big\{j, a_i a_{g+1-i}^{-1}, b_i b_{g+1-i}^{-1}, c_k c_{g-k}^{-1}, (c_r a_r a_{r+1})^2 e^{-1}; 1 \leqslant i \leqslant r,\ 1 \leqslant k \leqslant r-1\big\}$$

when $g = 2r$ and

$$\big\{j, a_i a_{g+1-i}^{-1}, b_i b_{g+1-i}^{-1}, c_k c_{g-k}^{-1}, (a_{r+1} c_r c_{r+1})^2 f^{-1};\ 1 \leqslant i, k \leqslant r\big\}$$

when $g = 2r + 1$.

4. McCarthy and Papadopoulos showed in [7] that the mapping class group of a closed orientable surface of genus greater or equal to three is generated by involutions. Here we prove an analogous thorem for non-orientable surfaces. This is the main result of this paper.

THEOREM 3. – *The mapping class group of a closed non-orientable surface of genus $g \geqslant 3$ is generated by involutions.*

Proof. – It suffices to show that each of the generators in Theorem 2 is a product of elements of order 2 in $S(\widetilde{X})/I(\widetilde{X})$.

Let $J_1 : \widetilde{X} \to \widetilde{X}$ be the symmetric homeomorphism $J_1(x, y, z) = (-x, y, z)$ and let j_1 denote its isotopy class. We have $j_1 a_i j_1 = a_i^{-1}$ for $1 \leqslant i \leqslant r$ and, since twists about disjoint circles commute,

$$j_1 a_i a_{g+1-i}^{-1} j_1 a_i a_{g+1-i}^{-1} = a_i^{-1} a_{g+1-i} a_i a_{g+1-i}^{-1} = 1.$$

Also $j_1^2 = 1$ and we have two elements of order 2, j_1 and $j_1 a_i a_{g+1-i}^{-1}$, whose product is $a_i a_{g+1-i}^{-1}$.

To prove that $b_i b_{g+1-i}^{-1}$ is a product of elements of order 2, it suffices to prove that it is conjugate in $S(\widetilde{X})/I(\widetilde{X})$ to $a_i a_{g+1-i}^{-1}$. The braid relations in the group $M(\widetilde{X})$ give:

$$\big(a_{g+1-i}^{-1} b_{g+1-i}^{-1} a_i b_i\big) a_i a_{g+1-i}^{-1} \big(a_{g+1-i}^{-1} b_{g+1-i}^{-1} a_i b_i\big)^{-1} = b_i b_{g+1-i}^{-1}.$$

Since $a_{g+1-i}^{-1} b_{g+1-i}^{-1} a_i b_i \in S(\widetilde{X})/I(\widetilde{X})$, we obtain that $a_i a_{g+1-i}^{-1}$ and $b_i b_{g+1-i}^{-1}$ are conjugate in $S(\widetilde{X})/I(\widetilde{X})$. Similarly, $c_k c_{g-k}^{-1}$ and $a_k a_{g+1-k}^{-1}$ are conjugate in $S(\widetilde{X})/I(\widetilde{X})$.

Let j_2 be the isotopy class of the symmetric homeomorphism $J_2(x, y, z) = (x, -y, z)$. We assume $g = 2r$ and set $y = (c_r a_r a_{r+1})^2 e^{-1}$. Using the relation $(a_{r+1} a_r c_r)^2 = (c_r a_r a_{r+1})^2$ (see [3]) we obtain:

$$j_2 y j_2 = \big(c_r^{-1} a_r^{-1} a_{r+1}^{-1}\big)^2 e = (a_{r+1} a_r c_r)^{-2} e = e(c_r a_r a_{r+1})^{-2} = y^{-1}.$$

Thus j_2 and $j_2 y$ both have order 2 and $y = j_2 j_2 y$. The same argument applies to the case $g = 2r + 1$. □

Note that the mapping class group of the projective plane ($g = 1$) is trivial. The mapping class group of the Klein bottle ($g = 2$) is known to be isomorphic to the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see [4]) so it is also generated by involution.

Our next theorem is an application of Theorem 3.

THEOREM 4. – *The moduli space of closed non-orientable Klein surfaces of genus* $g \geqslant 3$ *is simply-connected.*

Proof. – It is known [5] that each element of finite order in Mod_g has a fixed point in $\mathcal{T}_g$, so that, by Theorem 3, Mod_g is generated by elements which have fixed points. Also Mod_g is a properly discontinuous group of homeomorphisms of $\mathcal{T}_g$ and the stabiliser of each point of $\mathcal{T}_g$ is finite [5]. Thus, applying the result of Armstrong [2] we have that $\mathcal{T}_g/\mathrm{Mod}_g$ has trivial fundamental group. □

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