C. R. Acad. Sci. Paris, Ser. I 335 (2002) 993–996

Algèbres de Lie/*Lie Algebras*

Local cohomology and $\mathcal{D}$-affinity in positive characteristic

Masaki Kashiwara [a], **Niels Lauritzen** [b]

[a] Research Institute for Mathematical Sciences, Kyoto University, Japan

[b] Institut for matematiske fag, Aarhus Universitet, Århus, Denmark

Received 14 October 2002; accepted 23 October 2002

Note presented by Masaki Kashiwara.

Abstract We give an example of a $\mathcal{D}$-module on a Grassmann variety in positive characteristic with non-vanishing first cohomology group. This is a counterexample to $\mathcal{D}$-affinity and the Beilinson–Bernstein equivalence for flag manifolds in positive characteristic. ***To cite this article: M. Kashiwara, N. Lauritzen, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 993–996.***

Cohomologie locale et $\mathcal{D}$-affinité en caractéristique positive

Résumé On donne un exemple d'un $\mathcal{D}$-module sur une variété grassmannienne en caractéristique positive avec premier groupe de cohomologie non nul. On obtient ainsi un contre-exemple à la $\mathcal{D}$-affinité et à équivalence de Beilinson–Bernstein pour les variétés des drapeaux en caractéristique positive. ***Pour citer cet article : M. Kashiwara, N. Lauritzen, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 993–996.***

1. Introduction

Let k be a field. Consider the polynomial ring

$$R = k\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{bmatrix}$$

and $I \subset R$ the ideal generated by the three 2×2 minors

$$f_1 := \begin{vmatrix} X_{12} & X_{13} \\ X_{22} & X_{23} \end{vmatrix}, \quad f_2 := \begin{vmatrix} X_{13} & X_{11} \\ X_{23} & X_{21} \end{vmatrix} \quad \text{and} \quad f_3 := \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix}.$$

If k is a field of positive characteristic, the local cohomology modules $\mathrm{H}^j_I(R)$ vanish for $j > 2$ (see Chapitre III, Proposition 4.1 in [3]). However, if k is a field of characteristic zero, $\mathrm{H}^3_I(R)$ is non-vanishing (see Proposition 2.1 of this paper or Remark 3.13 in [5]).

Consider the Grassmann variety $X = \mathrm{Gr}(2, V)$ of 2-dimensional vector subspaces of a 5-dimensional vector space V over k. Let us take a two dimensional subspace W of V. Then the singularity R/I appears in the Schubert variety $Y \subset X$ of 2-dimensional subspaces E such that $\dim(E \cap W) \geqslant 1$. Therefore $\mathcal{H}^3_Y(\mathcal{O}_X)$ does not vanish in characteristic zero while it does vanish in positive characteristic.

E-mail addresses: masaki@kurims.kyoto-u.ac.jp (M. Kashiwara); niels@imf.au.dk (N. Lauritzen).

S1631-073X(02)02616-X/FLA

In this paper we show how this difference in the vanishing of local cohomology translates into a non-vanishing first cohomology group for the $\mathcal{D}_X$-module $\mathcal{H}^2_Y(\mathcal{O}_X)$ in positive characteristic.

Previous work of Haastert [4] showed the Beilinson–Bernstein equivalence [1] to hold for projective spaces and the flag manifold of SL_3 in positive characteristic. However, as we show in this paper, $\mathcal{D}$-affinity breaks down for the flag manifold of SL_5 in all positive characteristics. The Beilinson–Bernstein equivalence, therefore, does not carry over to flag manifolds in positive characteristic. On the other hand, for the sheaf of differential operators without divided powers, Bezrukavnikov, Mirković and Rumynin have recently proved that the Beilinson–Bernstein equivalence may be restored for large primes as an equivalence of bounded derived categories (see [2]).

2. Local cohomology

Keep the notation from Section 1. A topological proof of the following proposition is given in Section 5.

PROPOSITION 2.1. – $\mathrm{H}^3_I(R)$ *does not vanish in characteristic zero.*

COROLLARY 2.2. – $\mathcal{H}^3_Y(\mathcal{O}_X)$ *does not vanish in characteristic zero.*

The local to global spectral sequence

$$\mathrm{H}^p\big(X, \mathcal{H}^q_Y(\mathcal{O}_X)\big) \Rightarrow \mathrm{H}^{p+q}_Y(X, \mathcal{O}_X)$$

and $\mathcal{D}$-affinity in characteristic zero [1] implies

$$\mathrm{H}^3_Y(X, \mathcal{O}_X) = \Gamma\big(X, \mathcal{H}^3_Y(\mathcal{O}_X)\big) \neq 0.$$

On the other hand, if k is a field of positive characteristic, $\mathcal{H}^q_Y(\mathcal{O}_X) = 0$ if $q \neq 2$, since Y is a codimension two Cohen Macaulay subvariety of the smooth variety X ([3], Chapitre III, Proposition 4.1). This gives a totally different degeneration of the local to global spectral sequence. In the positive characteristic case we get

$$\mathrm{H}^p\big(X, \mathcal{H}^2_Y(\mathcal{O}_X)\big) \cong \mathrm{H}^{p+2}_Y(X, \mathcal{O}_X).$$

We will prove that $\mathrm{H}^3_Y(X, \mathcal{O}_X) \neq 0$ even if k is a field of positive characteristic. This will give the desired non-vanishing

$$\mathrm{H}^1\big(X, \mathcal{H}^2_Y(\mathcal{O}_X)\big) \neq 0$$

in positive characteristic.

3. Lifting to $\mathbb{Z}$

To deduce the non-vanishing of $\mathrm{H}^3_Y(X, \mathcal{O}_X)$ in positive characteristic, we need to compute the local cohomology over $\mathbb{Z}$ and proceed by base change. Flag manifolds and their Schubert varieties admit flat lifts to $\mathbb{Z}$-schemes. In this section $X_\mathbb{Z}$ and $Y_\mathbb{Z}$ will denote flat lifts of a flag manifold X and a Schubert variety $Y \subset X$ respectively. The local Grothendieck–Cousin complex (cf. [6], Section 8) of the structure sheaf $\mathcal{O}_{X_\mathbb{Z}}$

$$\mathcal{H}^0_{X_0/X_1}(\mathcal{O}_{X_\mathbb{Z}}) \to \mathcal{H}^1_{X_1/X_2}(\mathcal{O}_{X_\mathbb{Z}}) \to \cdots, \tag{1}$$

where X_i denotes the union of Schubert schemes of codimension i, is a resolution of $\mathcal{O}_{X_\mathbb{Z}}$, since $\mathcal{O}_{X_\mathbb{Z}}$ is Cohen Macaulay, codim $X_i \geqslant i$ and $X_i \setminus X_{i+1} \to X$ are affine morphisms for all i (see [6], Theorem 10.9). The sheaves in this resolution decompose into direct sums

$$\mathcal{H}^i_{X_i/X_{i+1}}(\mathcal{O}_{X_\mathbb{Z}}) = \bigoplus_{\mathrm{codim}(C)=i} \mathcal{H}^i_C(\mathcal{O}_{X_\mathbb{Z}})$$

of local cohomology sheaves $\mathcal{H}^i_C(\mathcal{O}_{X_\mathbb{Z}})$ with support in Bruhat cells C of codimension i. The degeneration of the local to global spectral sequence gives

$$\mathrm{H}^p_{Y_\mathbb{Z}}\big(X_\mathbb{Z}, \mathcal{H}^c_C(\mathcal{O}_{X_\mathbb{Z}})\big) = \mathrm{H}^{p+c}_{C\cap Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}),$$

To cite this article: M. Kashiwara, N. Lauritzen, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 993–996

since $\mathcal{H}^i_C(\mathcal{O}_{X_\mathbb{Z}}) = 0$ if $i \neq c = \operatorname{codim}(C)$. Since the scheme $X_i \setminus X_{i+1}$ is affine it follows that $\mathrm{H}^p_C(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) = 0$ if $p \neq \operatorname{codim}(C)$ ([6], Theorem 10.9). This shows that the resolution (1) is acyclic for the functor $\Gamma_{Y_\mathbb{Z}}$ and

$$\Gamma_{Y_\mathbb{Z}}\big(X_\mathbb{Z}, \mathcal{H}^c_C(\mathcal{O}_{X_\mathbb{Z}})\big) = \begin{cases} 0 & \text{if } C \not\subset Y_Z, \\ \mathrm{H}^c_C(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) & \text{if } C \subset Y_\mathbb{Z}, \end{cases}$$

where $c = \operatorname{codim}(C)$. Applying $\Gamma_{Y_\mathbb{Z}}(X_\mathbb{Z}, -)$ to (1) we get the complex

$$M^\bullet_Y : \mathrm{H}^c_{C_Y}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \to \bigoplus_{\operatorname{codim}(C)=c+1,\, C\subset Y_\mathbb{Z}} \mathrm{H}^{c+1}_C(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \to \cdots,$$

where c is the codimension of $Y_\mathbb{Z}$, C_Y is the open Bruhat cell in $Y_\mathbb{Z}$ and $\mathrm{H}^c_{C_Y}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ sits in degree c. Notice that $\mathrm{H}^i(M^\bullet_Y) = \mathrm{H}^i_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ and that $M^\bullet_Y$ is a complex of free abelian groups. In fact the individual entries $\mathrm{H}^i_C(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ are direct sums of weight spaces, which are finitely generated free abelian groups (cf. [6], Theorem 13.4). By weight spaces we mean eigenspaces for a fixed $\mathbb{Z}$-split torus T. The differentials in $M^\bullet_Y$ being T-equivariant, the complex $M^\bullet_Y$ is a direct sum of complexes of finitely generated free abelian groups. Since $\mathrm{H}^i(M^\bullet_Y) = \mathrm{H}^i_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$, one obtains the following lemma:

LEMMA 3.1. – *Every local cohomology group $\mathrm{H}^i_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ is a direct sum of finitely generated abelian groups. In the codimension c of $Y_\mathbb{Z}$ in $X_\mathbb{Z}$, $\mathrm{H}^c_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ is a free abelian group.*

4. The counterexample

For a field k, let us set $X_k = X_\mathbb{Z} \otimes k$ and $Y_k = Y_\mathbb{Z} \otimes k$. Then one has $\mathrm{H}^q_{Y_k}(X_k, \mathcal{O}_{X_k}) = \mathrm{H}^q_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}} \otimes k)$. Since $\mathcal{O}_{X_\mathbb{Z}}$ is flat over $\mathbb{Z}$, one has a spectral sequence

$$\mathrm{Tor}^\mathbb{Z}_{-p}\big(\mathrm{H}^q_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}), k\big) \Rightarrow \mathrm{H}^{p+q}_{Y_k}(X_k, \mathcal{O}_{X_k}).$$

This shows that the natural homomorphism $\mathrm{H}^i_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \otimes k \to \mathrm{H}^i_{Y_k}(X_k, \mathcal{O}_{X_k})$ is an injection, and it is an isomorphism if the field k is flat over $\mathbb{Z}$. In our example (cf. Section 1), one has $\mathrm{H}^3_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \otimes \mathbb{C} \cong \mathrm{H}^3_{Y_\mathbb{C}}(X_\mathbb{C}, \mathcal{O}_{X_\mathbb{C}}) \neq 0$.

By Lemma 3.1, the cohomology $\mathrm{H}^3_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ must contain $\mathbb{Z}$ as a direct summand. Therefore the injection $\mathrm{H}^3_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \otimes k \to \mathrm{H}^3_{Y_k}(X_k, \mathcal{O}_{X_k})$ shows that $\mathrm{H}^3_{Y_k}(X_k, \mathcal{O}_{X_k})$ is non-vanishing for any field k of positive characteristic. Since $\mathrm{H}^3_{Y_k}(X_k, \mathcal{O}_{X_k}) \cong \mathrm{H}^1(X_k, \mathcal{H}^2_{Y_k}(\mathcal{O}_{X_k}))$, one obtains the following result.

PROPOSITION 4.1. – $\mathrm{H}^1(X_k, \mathcal{H}^2_{Y_k}(\mathcal{O}_{X_k})) \neq 0$ *if k is of positive characteristic.*

5. Proof of non-vanishing of $\mathrm{H}^3_I(R)$

In this section, we shall give a topological proof of Proposition 2.1. We may assume that the base field is the complex number field $\mathbb{C}$.

The local cohomologies $\mathrm{H}^*_I(R)$ are the cohomology groups of the complex

$$R \to R\big[f_1^{-1}\big] \oplus R\big[f_2^{-1}\big] \oplus R\big[f_3^{-1}\big] \to R\big[(f_1 f_2)^{-1}\big] \oplus R\big[(f_2 f_3)^{-1}\big] \oplus R\big[(f_1 f_3)^{-1}\big] \to R\big[(f_1 f_2 f_3)^{-1}\big].$$

Hence one has

$$\mathrm{H}^3_I(R) = \frac{R[(f_1 f_2 f_3)^{-1}]}{R[(f_1 f_2)^{-1}] + R[(f_2 f_3)^{-1}] + R[(f_1 f_3)^{-1}]}.$$

In order to prove the non-vanishing of $\mathrm{H}^3_I(R)$, it is enough to show

$$\frac{1}{f_1 f_2 f_3} \notin R\big[(f_1 f_2)^{-1}\big] + R\big[(f_2 f_3)^{-1}\big] + R\big[(f_1 f_3)^{-1}\big]. \tag{2}$$

Consider the 6-cycle

$$\gamma = \left\{ \begin{pmatrix} -t_2u + t_3\bar{v} & u & -t_1\bar{v} \\ -t_2v - t_3\bar{u} & v & t_1\bar{u} \end{pmatrix};\ |t_1| = |t_2| = |t_3| = 1,\ |u|^2 + |v|^2 = 1 \right\}$$
$$= \left\{ k \begin{pmatrix} -t_2 & 1 & 0 \\ -t_3 & 0 & t_1 \end{pmatrix};\ |t_1| = |t_2| = |t_3| = 1,\ k = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \in \mathrm{SU}(2) \right\}$$

in $X \setminus (f_1 f_2 f_3)^{-1}(0)$, where $X = \mathrm{Spec}(R) \cong \mathbb{C}^6$. Then on γ one has $f_1 = t_1$, $f_2 = t_1t_2$ and $f_3 = t_3$. Set $\omega = \bigwedge \mathrm{d}X_{ij}$. Then one has $\omega = t_1\, \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}t_3\, \theta$ on γ, where θ is a non-zero invariant form on SU(2). Therefore one has

$$\int_\gamma \frac{\omega}{f_1 f_2 f_3} = \int_\gamma \frac{\mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}t_3\, \theta}{t_1t_2t_3} \neq 0.$$

Hence, in order to show (2), it is enough to prove that

$$\int_\gamma \varphi\omega = 0 \tag{3}$$

for any $\varphi \in R[(f_1 f_2)^{-1}] + R[(f_2 f_3)^{-1}] + R[(f_1 f_3)^{-1}]$.

For $\varphi \in R[(f_1 f_2)^{-1}]$, Eq. (3) holds because we can shrink the cycle γ by $|t_3| = \lambda$ from $\lambda = 1$ to $\lambda = 0$. For $\varphi \in R[(f_1 f_3)^{-1}]$, Eq. (3) holds because we can shrink the cycle γ by $|t_2| = \lambda$ from $\lambda = 1$ to $\lambda = 0$.

Let us show (3) for $\varphi \in R[(f_2 f_3)^{-1}]$. Let us deform the cycle γ by

$$\gamma_\lambda = \left\{ k \begin{pmatrix} -(1-\lambda)t_2 & 1 & -\lambda t_1t_2t_3^{-1} \\ -t_3 & 0 & t_1 \end{pmatrix};\ |t_1| = |t_2| = |t_3| = 1,\ k \in \mathrm{SU}(2) \right\}.$$

Note that the values of f_1, f_2 and f_3 do not change under this deformation. Hence γ_λ is a cycle in $X \setminus (f_1 f_2 f_3)^{-1}(0)$. One has

$$\gamma_1 = \left\{ k \begin{pmatrix} 0 & 1 & -t_1t_2t_3^{-1} \\ -t_3 & 0 & t_1 \end{pmatrix};\ |t_1| = |t_2| = |t_3| = 1,\ k \in \mathrm{SU}(2) \right\}$$
$$= \left\{ k \begin{pmatrix} 0 & 1 & -t_2 \\ -t_3 & 0 & t_1 \end{pmatrix};\ |t_1| = |t_2| = |t_3| = 1,\ k \in \mathrm{SU}(2) \right\}.$$

In the last coordinates of γ_1, one has $f_2 = t_2t_3$ and $f_3 = t_3$. Hence, for $\varphi \in R[(f_2 f_3)^{-1}]$,

$$\int_\gamma \varphi\omega = \int_{\gamma_1} \varphi\omega$$

vanishes because we can shrink the cycle γ_1 by $|t_1| = \lambda$ from $\lambda = 1$ to $\lambda = 0$.

Remark 1. – Although we do not give a proof here, $\mathrm{H}^3_I(R)$ is isomorphic to $\mathrm{H}^6_J(R)$ as a D-module. Here J is the defining ideal of the origin.

References

[1] A. Beilinson, J. Bernstein, Localisation de $\mathfrak{g}$-modules, C. R. Acad. Sci. Paris 292 (1981) 15–18.

[2] R. Bezrukavnikov, I. Mirković, D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic, Preprint, math.RT/0205144.

[3] C. Peskine, L. Szpiro, Dimension projective finie et cohomologie locale, Inst. Hautes Études Sci. Publ. Math. 42 (1973) 47–119.

[4] B. Haastert, Über Differentialoperatoren und $\mathcal{D}$-Moduln in positiver Characteristik, Manuscripta Math. 58 (1987) 385–415.

[5] C. Huneke, G. Lyubeznik, On the vanishing of local cohomology modules, Invent. Math. 102 (1990) 73–95.

[6] G. Kempf, The Grothendieck–Cousin complex of an induced representation, Adv. Math. 29 (1978) 310–396.