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Étale covers of affine spaces in positive characteristic

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Abstract We prove that every projective, geometrically reduced scheme of dimension n over an infinite field k of positive characteristic admits a finite morphism over some finite radicial extension k' of k to projective n-space, étale away from the hyperplane H at infinity, which maps a chosen Weil divisor into H and a chosen smooth geometric point of X not on the divisor to some point not in H. ***To cite this article: K.S. Kedlaya, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 921–926.***

Revêtements étales des espaces affines en caractéristique positive

Résumé Nous prouvons que tout schéma projectif, géométriquement réduit de dimension n sur un corps infini k de caractéristique positive admet un morphisme fini aprés extension finie radicielle k' de k, vers l'espace projectif de dimension n, étale sauf sur l'hyperplan H a l'infini, qui envoie dans H un diviseur de Weil choisi et un point géométrique lisse choisi de X en-dehors du diviseur sur un point en-dehors de H. ***Pour citer cet article : K.S. Kedlaya, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 921–926.***

Version française abrégée

Un théorème celèbre de Belyĭ [1] affirme qu'une courbe lisse, projective, et irreductible sur les nombres complexes peut être définée sur un corps de nombres si et seulement si elle admet un morphisme vers $\mathbb{P}^1$ ramifié en au plus trois points. En caractéristique positive, les revêtements de $\mathbb{P}^1$ avec peu de ramification sont plus fréquents : toute courbe sur un corps infini de caractéristique $p > 0$ admet un morphisme vers $\mathbb{P}^1$ ramifié seulement au-dessus d'un point ! Ce résultat est à la fois facile à prouver et d'une utilité suprenante, particulièrement quand on a besoin de remplacer un problème sur une courbe compliquée par « image directe » sur la droite affine. Confer [3] pour la preuve de ce résultat et une application du type indiqué.

Dans cette Note, nous généralisons le résultat en caractéristique positive aux schémas de dimension $\geqslant 1$. On peut considérer ce résultat comme une analogue en caractéristique positive d'un theoreme de Bogomolov–Pantev [2].

THÉORÈME 1. – *Soit X un schéma séparé, géométriquement réduit, projectif, purement de dimension n sur un corps infini k de caractéristique $p > 0$. Soit D un diviseur de Weil sur X, et soit x un point lisse*

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de $X(k^{\mathrm{sep}})$*, non contenu dans* D*. Alors il existe une extension finie radicielle* k' *de* k *et un morphisme fini* $f : X \to \mathbb{P}^n$ *de* k'*-schémas satisfaisant les conditions suivantes* :

(1) *le morphisme* f *est étale sauf au-dessus de l'hyperplan* $H \subseteq \mathbb{P}^n$ *à l'infini* ;
(2) *l'image* $f(D)$ *est contenue dans* H ;
(3) *l'image* $f(x)$ *n'est pas contenue dans* H*.*

La restriction que k soit infini est nécessaire par accomplir certains constructions de façon « générique » ; le passage à une extension radicielle est nécessaire par assurer que les sous-schémas réduits de certains diviseurs de ramification soient géométriquement réduits.

COROLLAIRE 2. – *Soit* X *un schéma séparé, géométriquement réduit, et de type fini, purement de dimension* n*, sur un corps infini* k*, et soit* x *un point lisse de* $X(k^{\mathrm{sep}})$*. Alors il existe une extension finie radicielle* k' *de* k *et un morphisme fini et étale* $f : U \to \mathbb{A}^n$ *sur* k'*, où* U *est un sous-schéma ouvert dense de* X *contenant* x*.*

Le corollaire suffit pour bien des applications : par exemple, la preuve de la finitude de la cohomologie rigide de Berthelot avec coefficients dans un F-isocrystal surconvergent [4] utilise seulement le corollaire.

La démonstration du théorème procéde en trois étapes. Premièrement, on projete X sur $\mathbb{P}^n$ par une normalisation de Noether. Deuxièmement, on simplifie le diviseur de ramification, en utilisant des morphismes produit de polynômes additifs ; le diviseur restant et l'union de $n+1$ hyperplans en position générale. Troisièmement, on utilise une variation du morphisme d'Abhyankar (qui présente le groupe multiplicatif $\mathbb{G}_m$ comme un revêtement étale de le droite affine $\mathbb{A}^1$) pour réduire le diviseur de ramification à un seul hyperplan.

1. Introduction

A celebrated theorem of Belyĭ [1] asserts that a smooth, projective, irreducible curve over the complex numbers can be defined over a number field if and only if it admits a map to $\mathbb{P}^1$ ramified over at most three points. In positive characteristic, covers of $\mathbb{P}^1$ with even less ramification are far more prevalent: every curve over an infinite field of characteristic $p > 0$ admits a map to $\mathbb{P}^1$ ramified over only one point! This assertion is both easy to prove and surprisingly useful, especially when one wants to "push forward" some problem from a complicated curve to a simple curve like the affine line. See [3] for both the proof of the assertion (on which this Note is ultimately based) and an application of the indicated type.

In this Note, we generalize the positive characteristic assertion to higher dimensional schemes as follows. This result may be viewed as a positive characteristic analogue of a theorem of Bogomolov–Pantev [2].

THEOREM 1. – *Let* X *be a geometrically reduced, projective scheme of pure dimension* n *over an infinite field* k *of characteristic* $p > 0$*. Let* D *be a Weil divisor of* X *and let* x *be a smooth point of* $X(k^{\mathrm{sep}})$ *not contained in* D*. Then there exists a finite radicial extension* k' *of* k *and a finite morphism* $f : X \to \mathbb{P}^n$ *of* k'*-schemes satisfying the following conditions*:

(1) *the morphism* f *is étale away from the hyperplane* $H \subseteq \mathbb{P}^n$ *at infinity*;
(2) *the image* $f(D)$ *is contained in* H;
(3) *the image* $f(x)$ *is not contained in* H*.*

It may be possible to eliminate the hypothesis that k is infinite (which occurs because of several "generic" constructions in the proof) or the need to pass to a finite radicial extension k'.

COROLLARY 2. – *Let* X *be a separated, geometrically reduced scheme of finite type over an infinite field* k*, of pure dimension* n*, and let* x *be a smooth point of* $X(k^{\mathrm{sep}})$*. Then there exists a finite radicial extension* k' *of* k *and a finite étale morphism* $f : U \to \mathbb{A}^n$ *over* k'*, for* U *some open dense subscheme of* X *containing* x*.*

By Noetherian induction, the smooth locus of X can thus be covered with open (necessarily affine) subsets which are finite étale covers of $\mathbb{A}^n$.

Note that the theorem is strictly stronger than the corollary; from the corollary, one only gets a rational map from X to $\mathbb{P}^n$. However, in some cases it may be the corollary that is most directly useful, again when one needs to "push forward" a problem to a simpler space via an étale map, but only on an open dense subset of the original space. One example of this situation is the author's proof of finite dimensionality of rigid cohomology with coefficients in an overconvergent F-isocrystal [4].

2. Proof of the theorem

To prove the theorem, we will need to string together a chain of carefully chosen maps. To facilitate this, we make the following definitions. A *good triple* over a finite radicial extension k' of k will always mean a triple (Y, E, y), where Y is a separated, projective, geometrically reduced scheme of pure dimension n over k', E is a Weil divisor of Y defined over k', and y is a point of $Y((k')^{\mathrm{sep}})$ not contained in E. Note that this notion is stable under extending k'. Given two good triples (Y_1, E_1, y_1) and (Y_2, E_2, y_2) over k', a *good morphism* over k' will be a finite morphism $f : Y_1 \to Y_2$ of k'-schemes, with $f(E_1) \subseteq E_2$, $f(y_1) = y_2$, and f étale on $Y_2 \setminus E_2$.

In this language, the given triple (X, D, x) is good, and the problem is to find a chain of good morphisms leading from (X, D, x) to $(\mathbb{P}^n, H, z)$ for some $z \notin H$. We construct this chain in three steps.

Reminder. – The assertion "property P holds for the generic Y" means that property P holds for all Y in an open dense subset of the natural parameter space of all objects Y. In particular, this type of assertion is stable under conjunction on property P. Moreover, since k is infinite, if property P holds for the generic Y, then it actually holds for some choice of Y (and in fact for infinitely many choices) defined over k.

2.1. Step 1: Noether normalization

For our first step, we construct a good morphism $\pi : (X, D, x) \to (\mathbb{P}^n, D_0, x_0)$ over some k_0 by Noether normalization. Choose a projective embedding $g : X \to \mathbb{P}^m$ of X. For a generic $(m-n-1)$-plane P in $\mathbb{P}^m$, the map $\pi : X \to \mathbb{P}^n$ induced by projection away from P is finite and has the following additional properties:

(a) the image $\pi(x)$ is not contained in $\pi(D)$, that is, P does not meet the join J of x and D. That is because $\dim J + \dim P = n + (m-n-1) < m$;

(b) the morphism π is étale over $\pi(x)$. This follows from Bertini's theorem and the fact that a generic $(m-n)$-plane through x is the intersection of $n-m$ generic hyperplanes: the intersection of X with one generic hyperplane is smooth at x, the intersection of the result with a second generic hyperplane is again smooth at x, and so on, until the intersection of X with the $(m-n)$-plane is smooth at x and hence reduced.

Fixing a choice of P, take $x_0 = \pi(x)$. After replacing k by a suitable finite radicial extension k_0, the reduced subscheme of the union of $\pi(D)$ with the branch locus of π will be geometrically reduced; call this reduced subscheme D_0.

We have now eliminated all of the geometry of the ambient scheme X from the discussion; the rest of the argument takes place within the projective space $\mathbb{P}^n$.

2.2. Step 2: Additive polynomials

In this step, we construct an increasing sequence of finite radicial extensions k_i of k, a sequence of good triples $(\mathbb{P}^n, x_i, D_i)$ over k_i for $i = 0, \ldots, n$, starting with the good triple $(\mathbb{P}^n, x_0, D_0)$ from the previous step, and a sequence of good morphisms $f_i : (\mathbb{P}^n, x_i, D_i) \to (\mathbb{P}^n, x_{i+1}, D_{i+1})$ over k_{i+1}, such that D_i is the union of i hyperplanes H_{ij} $(j = 0, \ldots, i-1)$ in general position (that is, whose mutual intersection has codimension i) with the join C_i of a geometrically reduced hypersurface within $\bigcap H_{ij}$ with a plane of dimension i in $\mathbb{P}^n$ not meeting $\bigcap_j H_{ij}$. In particular, for $i = 0$, C_i is simply D_i itself, while for $i = n$, a plane of codimension n is a point and a hypersurface therein is empty, so D_n will be the union of $n + 1$ hyperplanes meeting transversely.

Before proceeding to the construction, we recall a bit of algebra peculiar to positive characteristic; the proof is standard, so we omit it.

LEMMA 3. – *Let R be a ring of prime characteristic $p > 0$. Then for any monic polynomial $P \in R[t]$ of degree m, there is a canonical multiple Q of P having the form*

$$Q(t) = \sum_{i=0}^{m} r_i t^{p^i}$$

for some $r_i \in R$ with $r_m = 1$. If R is a domain with fraction field K and P has distinct roots in K^{sep} which are linearly independent over $\mathbb{F}_p$, then r_0 is nonzero. Moreover, if $R = k[x_1, \dots, x_l]$ and P is homogeneous as a polynomial in $x_1, \dots, x_l, t$, then so is Q.

A polynomial of the form prescribed for Q is called *additive*, since such polynomials are precisely those for which $Q(t + u) = Q(t) + Q(u)$ identically.

We first outline the construction of f_i given x_i and D_i, then record the geometric conditions that must be satisfied for the construction to go through. The construction will depend on a choice of homogeneous coordinates $z_0, \dots, z_n$ (i.e., a basis for the space of sections of $\mathcal{O}(1)$) such that H_{ij} is the zero locus of z_j and the defining equation P_i of C_i depends only on $z_i, \dots, z_n$. This condition determines each of $z_0, \dots, z_{i-1}$ up to a scalar, and determines the span of $z_i, \dots, z_n$. Thus the eligible coordinate systems are parametrized by $\mathbb{G}_m^i \times \mathrm{GL}(n - i + 1)$, an irreducible variety; we will ultimately show that the geometric conditions are satisfied on an open and nonempty, so dense, subset of the parameter variety.

Regard P_i as a polynomial in z_i whose coefficients are polynomials in $z_{i+1}, \dots, z_n$. If P_i has the same degree in z_i alone as its total degree in $z_i, \dots, z_n$, then some scalar multiple of P_i is monic in z_i. In that case, let Q_i be the multiple of (that scalar multiple of) P_i produced by the previous lemma, and put $d_i = \deg(Q_i)$. Now define the map $f_i : \mathbb{P}^n \to \mathbb{P}^n$ by sending $(z_0 : \cdots : z_n)$ to $(w_0 : \cdots : w_n)$, where

$$w_j = \begin{cases} z_j^{d_i} - z_j z_n^{d_i - 1}, & j \neq i, n, \\ Q_i(z_i, \dots, z_n), & j = i, \\ z_n^{d_i}, & j = n, \end{cases}$$

and take $x_{i+1} = f_i(x_i)$, $H_{(i+1)j}$ to be the zero locus of w_j for $j = 0, \dots, i$, k_{i+1} to be a finite radicial extension of k_i over which the reduced subscheme of the zero locus of the degree 1 coefficient of Q_i is geometrically reduced, and C_{i+1} to be that reduced subscheme. In particular, C_{i+1} is the zero locus of a polynomial depending only on $z_{i+1}, \dots, z_n$.

For f_i to be a regular map, the w_j must have no common zeroes. In that case, the nonétale locus of f_i is contained in the zero locus of z_n times the coefficient of Q_i in degree 1. In short, the construction gives what we want provided that the following conditions hold.

(a) The degree of P_i as a polynomial in z_i alone is equal to its total degree in $z_i, \dots, z_n$.
(b) The values of z_n and $z_j^{d_i} - z_j z_n^{d_i - 1}$ at x_i are nonzero for $j \neq i, n$.
(c) The constant coefficient of Q_i is nonzero.
(d) The value of Q_i at x_i is nonzero.

Each of these is clearly an open condition on the parameter variety of coordinate systems $z_0, \dots, z_n$. We conclude the construction by verifying that each of conditions (a)–(d) is not identically violated. Then each condition holds on an open dense subset of the parameter variety; since k is infinite, the intersection of these open dense subsets contains infinitely many k-rational points, any one of which yields a satisfactory choice of f_i.

Condition (a) is violated if and only if P_i vanishes identically on the plane with $z_j = 0$ for $j > i$, so this condition does not hold for all coordinate systems. Similarly (b) is not identically violated. To check (c) and (d), it suffices to work in the projection from $\bigcap H_{ij}$. In the image of this projection, draw the line through the image of x_i and the point with $z_0 = \cdots = z_{n-1} = 0$, and choose an identification of this line with $\mathbb{P}^1$ in which the latter point becomes ∞. For a generic choice of the line, the intersections of C_i with the line, plus x_i, will be distinct and defined over k_i^{sep} because C_i is geometrically reduced. Then (c) is satisfied if the intersections of C_i with the line are identified with a set of elements of k_i^{sep} which are linearly independent over $\mathbb{F}_p$ (and in that case $d_i = p^{\deg P_i}$), and (d) is satisfied if the same holds after including x_i as well.

We now turn the tables, regarding the line as fixed and varying the coordinate system over k_i, under the constraint that the point with $z_0 = \cdots = z_{n-1} = 0$ remains on the line. As we do this, the collection of elements of k_i^{sep} that we wrote down previously is moved around by linear fractional transformations over k_i, and by the following lemma, for a generic choice of this transformation, the collection of elements becomes linearly independent over $\mathbb{F}_p$.

LEMMA 4. – *Let* $\{r_1, \ldots, r_m\}$ *be a finite subset of* k_i^{sep} *stable under* $\mathrm{Gal}(k_i^{\mathrm{sep}}/k_i)$. *Then for a generic choice of* $a, b, c, d \in k_i$ *(i.e., away from a Zariski closed subset of* $\mathbb{A}^4_{k_i}$*), if we set* $\tau(x) = (a + bx)/(c + dx)$*, then*

$$h_1\tau(r_1) + \cdots + h_m\tau(r_m) \neq 0$$

for any $h_1, \ldots, h_m \in \mathbb{F}_p$ *not all zero.*

Proof. – Let K be the field generated over k_i by $r_1, \ldots, r_m$. Then the condition $h_1\tau(r_1) + \cdots + h_m\tau(r_m) \neq 0$ can be written as an algebraic condition on a, b, c, d over k_i by decomposing each term over a basis for K over k_i. Thus it suffices to check that this condition does not hold identically. In fact, it suffices to check $h_1\tau(r_1) + \cdots + h_m\tau(r_m) \neq 0$ separately for each choice of $h_1, \ldots, h_m \in \{0, \ldots, p-1\}$, since there are finitely many such choices; moreover, it is enough to check under the additional restriction $a = b = 0$ and $d = 1$. In that case, the expression in question becomes

$$\frac{h_1}{c + x_1} + \cdots + \frac{h_m}{c + x_m} = \frac{R'(c)}{R(c)},$$

where $R(x) = \prod_j (x + x_j)^{h_j}$. Since $R(x)$ is not a p-th power, its derivative does not vanish identically. Thus the expression does not vanish identically over all choices of a, b, c, d in k_i, as desired. □

Thus (c) and (d) hold for some coordinate system, completing the verification of the necessary conditions for the construction of f_i.

2.3. Step 3: The Abhyankar map

For the third step, we must construct a good morphism $f_n : (\mathbb{P}^n, x_n, D_n) \to (\mathbb{P}^n, x_{n+1}, H)$ for some x_{n+1}, where D_n is the union of n transverse hyperplanes. We explicitly construct this morphism by writing down polynomials g_i in the variables $z_0, \ldots, z_n$, for $i = 0, \ldots, n$, as follows. For each $(i+1)$-element subset $I = \{j_0, \ldots, j_i\}$ of $\{0, \ldots, n\}$, with $j_0 < \cdots < j_i$, define

$$m_I = z_{j_0}^{1+p+\cdots+p^{n-i}} z_{j_1}^{p^{n-i+1}} \cdots z_{j_i}^{p^n}$$

and let g_i be the sum of the m_I over all $(i+1)$-element subsets I. For example, when $n = 2$, we have

$$\begin{aligned}g_0 &= z_0^{p^2+p+1} + z_1^{p^2+p+1} + z_2^{p^2+p+1},\\ g_1 &= z_0^{p+1}z_1^{p^2} + z_0^{p+1}z_2^{p^2} + z_1^{p+1}z_2^{p^2},\\ g_2 &= z_0 z_1^p z_2^{p^2}.\end{aligned}$$

Let us observe some facts about the g_i. First, they are all homogeneous of degree $1 + p + \cdots + p^n$. Second, they have no common zero except $z_0 = \cdots = z_n = 0$, by the same argument as for the elementary symmetric functions: if $g_n = 0$, then one of the z_i must be zero; in that case, if $g_{n-1} = 0$, then another of the z_i must be zero, and so on. These two facts allow us to define a morphism $f_n : \mathbb{P}^n \to \mathbb{P}^n$ by the formula

$$(z_0 : \cdots : z_n) \mapsto (g_0 : \cdots : g_n).$$

Third, note that for $z_0, \ldots, z_n$ all nonzero, the differentials $\mathrm{d}g_0, \ldots, \mathrm{d}g_n$ are linearly independent. Namely, $\mathrm{d}g_n$ is a nonzero multiple of $\mathrm{d}z_0$; $\mathrm{d}g_{n-1}$ is a nonzero multiple of $\mathrm{d}z_1$ plus a multiple of $\mathrm{d}z_0$; $\mathrm{d}g_{n-2}$ is a nonzero multiple of $\mathrm{d}z_2$ plus a linear combination of $\mathrm{d}z_0$ and $\mathrm{d}z_1$; and so on. This means that f_n is étale away from the zero locus of $z_0 \cdots z_n$, i.e., the zero locus of g_n.

In passing, we note that the case $n = 1$ of this construction yields what is commonly called the Abhyankar map, which expresses the affine line minus a point as an étale cover of the full affine line. It seems a fitting tribute to Abhyankar's work to bestow the same name on this higher-dimensional analogue.

To conclude, if we set $x_{n+1} = f_n(x_n)$, and H equal to the hyperplane $z_n = 0$, the map f_n gives a good morphism from $(\mathbb{P}^n, x_n, D_n)$ to $(\mathbb{P}^n, x_{n+1}, H)$. Stringing together the good morphisms $f_0, \ldots, f_n$ yields a good morphism from (X, x, D) to $(\mathbb{P}^n, x_{n+1}, H)$, completing the proof of the theorem.

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