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An anomaly formula for Ray–Singer metrics on manifolds with boundary

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Abstract We establish an anomaly formula for Ray–Singer metrics defined by a Hermitian metric on a flat vector bundle over a Riemannian manifold with boundary. We do not assume that the Hermitian metric on the flat vector bundle is flat, nor that the Riemannian metric has product structure near the boundary. ***To cite this article: J. Brüning, X. Ma, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 603–608.***

Formules d'anomalie pour les métriques de Ray–Singer sur les variétés à bord

Résumé On annonce une formule d'anomalie pour les métriques de Ray–Singer d'un fibré plat F sur une variété à bord X . On ne suppose ni que la métrique sur F est plate, ni que la métrique sur X a une structure produit près du bord. ***Pour citer cet article : J. Brüning, X. Ma, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 603–608.***

Version française abrégée

Soit X une variété compacte à bord Y. Soit (F, ∇^F) un fibré vectoriel complexe plat sur X. Soit g^{TX} une métrique riemannienne sur TX, soit h^F une métrique hermitienne sur F.

Soit $H^\bullet(X, F) = \bigoplus_{p=0}^m H^p(X, F)$ la cohomologie de de Rham absolue de X à coefficients dans F. La métrique de Ray–Singer sur la droite complexe $\det H^\bullet(X, F) = \bigotimes_{p=0}^m (\det H^p(X,\ F))^{(-1)^p}$ est le produit de la métrique L^2 standard sur $\det H^\bullet(X, F)$ et de la torsion analytique de Ray–Singer [14].

Dans cette Note, on annonce une formule d'anomalie pour les métriques de Ray–Singer, qui généralise le résultat correspondant pour les variétés sans bord [2, Théorème 0.1]. On ne suppose ni que la métrique sur F est plate, et ni que la métrique sur X a une structure produit près du bord.

Dans notre formule, la contribution du bord est obtenue à partir de la solution fondamentale d'un problème modèle sur $\mathbb{R}^{m-1} \times \mathbb{R}_+$ avec condition de bord.

Les résultats annoncés dans cette Note sont démontrés dans [5].

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0. Introduction

Let X be a m-dimensional compact smooth manifold with boundary $\partial X = Y$, and let F be a flat complex vector bundle over X, with flat connection ∇^F. We denote by $H^\bullet(X,F) = \bigoplus_{p=0}^m H^p(X,\ F)$ the de Rham cohomology of X with coefficients in F with absolute boundary conditions. If E is a finite dimensional vector space, let $\det E := \Lambda^{\max} E$, and denote by $(\det E)^{-1} := \det E^*$ the dual line. The complex line $\det H^\bullet(X,F) = \bigotimes_{p=0}^m (\det H^p(X,F))^{(-1)^p}$ is the determinant of the cohomology of F.

Choose a Hermitian metric, h^F, on F and a smooth Riemannian metric, g^{TX}, on TX. By Hodge–de Rham theory, the de Rham cohomology $H^\bullet(X,F)$ is canonically isomorphic to the kernel of the associated Laplacian. Hence the chosen metrics induce a canonical L^2-metric, $h^{H^\bullet(X,F)}$, on $H^\bullet(X,F)$. Then the Ray–Singer metric, $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F)}$, on $\det H^\bullet(X,F)$ is defined as the product of the metric induced on $\det H^\bullet(X,F)$ by $h^{H^\bullet(X,F)}$ with the Ray–Singer analytic torsion [14], see also Definition 1.2.

If $Y = \emptyset$ and h^F is flat, $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F)}$ does not depend on g^{TX}. The Cheeger–Müller theorem [6,12] tells us that, in this case, the Ray–Singer metric can be identified with the Reidemeister metric, which is a topological invariant of the flat bundle F. Müller [13] extended his result to the case where $m = \dim X$ is odd and only the metric induced on $\det F$ is required to be flat. Bismut and Zhang [2] generalized this discussion to arbitrary flat vector bundles with arbitrary metrics and showed that in even dimension, the independence ceases to hold. There are also various extensions to the equivariant case, cf. [9,10,3].

Now consider X with $Y \neq \emptyset$. This case was studied in [9] and [10] under the assumption that h^F is flat and that g^{TX} is product near the boundary. Dai and Fang [8] were the first to study this problem with flat h^F but without assuming a product structure for g^{TX} near Y, by methods completely different from ours.

In this Note, we announce an anomaly formula for Ray–Singer metrics in the general case, allowing arbitrary Riemannian metrics on X and arbitrary Hermitian metrics on F. Our method also leads to a local Gauss–Bonnet–Chern theorem [7] for manifolds with boundary. The full details of our results are given in [5].

1. Analytic torsion for manifolds with boundary

Denote by $\Omega(X,F) := \bigoplus_{p=0}^m \Omega^p(X,F) := \bigoplus_{p=0}^m C^\infty(X, \Lambda^p(T^*X)\otimes F)$ the space of smooth differential forms on X with values in F. The flat connection extends naturally to a differential, d^F, on $\Omega(X,F)$. The metrics g^{TX}, h^F induce a Hermitian metric $\langle,\rangle_{\Lambda(T^*X)\otimes F}$ on $\Lambda(T^*X)\otimes F$. Let dv_X be the Riemannian volume element on (TX, g^{TX}). Let $o(TX)$ be the orientation bundle of TX, which is a flat real line bundle on X [4, p. 88]; then we can view dv_X as a section of $\Lambda^m(T^*X)\otimes o(TX)$. We define the Hermitian product on $\Omega(X,F)$ by $(\sigma,\sigma') := \int_X \langle\sigma,\sigma'\rangle_{\Lambda(T^*X)\otimes F}\,\mathrm{d}v_X$, for $\sigma,\sigma' \in \Omega(X,F)$. The Hilbert space obtained by completion is denoted by $\mathrm{L}^2(X,F)$.

We consider d^F as an unbounded operator in $\mathrm{L}^2(X,F)$ with domain $\Omega_0(X,F) := \{\sigma \in \Omega(X,F);\ \operatorname{supp}\sigma \cap Y = \emptyset\}$. The adjoint operator d^{F*} is also defined on $\Omega_0(X,F)$, and so is $D := d^F + d^{F*}$.

Next we define self-adjoint extensions of D by elliptic boundary conditions. We use the metric on X to identify the normal bundle $N_{Y/X}$ to Y in X with the orthogonal complement of TY in $TX|Y$. Denote by $e_{\mathfrak{n}}$ the inward pointing unit normal vector field along Y. Then we put, with $i(.)$ interior multiplication,

$$\begin{aligned}
&\Omega_a^p(X,F) := \bigl\{\sigma \in \Omega^p(X,F);\ i(e_{\mathfrak{n}})\sigma = i(e_{\mathfrak{n}})\bigl(d^F\sigma\bigr) = 0 \text{ on } Y\bigr\},\\
&D_a := D|\Omega_a(X,F) := D|\bigoplus_{p=0}^m \Omega_a^p(X,F), \qquad H_a^p(X,F) := \ker D_a \cap \Omega_a^p(X,F).
\end{aligned} \tag{1}$$

The operator D_a is essentially self-adjoint and we denote its closure also by D_a. By the de Rham–Hodge theorem for manifolds with boundary, $H_a^p(X,F)$ is canonically isomorphic to $H^p(X,F)$. We denote by $h^{H^\bullet(X,F)}$ the L^2-metric induced on $H^\bullet(X,F)$ by this isomorphism, and by $|\ |^{L^2}_{\det H^\bullet(X,F)}$ the corresponding metric on $\det H^\bullet(X,F)$.

Let P_a be the orthogonal projection in $\mathrm{L}^2(X,F)$ onto $H_a^\bullet(X,F)$ with $P_a^\perp := 1 - P_a$, and let N be the number operator on $\Lambda(T^*X)\otimes F$, which is multiplication by p on $\Lambda^p(T^*X)\otimes F$. Let $\exp(-tD_a^2)$ be the heat semi-group of D_a^2.

DEFINITION 1.1. – For $s\in\mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}\dim X$, set

$$\begin{aligned}\theta_a^F(s) &:= -\mathrm{Tr}_s\big[N\big(D_a^2\big)^{-s}P_a^\perp\big] := \textstyle\sum_{p=0}^m (-1)^{p+1} p\,\mathrm{Tr}_{\Omega_a^p(X,F)}\big[\big(D_a^2\big)^{-s}P_a^\perp\big]\\ &= -\frac{1}{\Gamma(s)}\int_0^{+\infty} t^s\,\mathrm{Tr}_s\big[N\exp\big(-tD_a^2\big)P_a^\perp\big]\frac{\mathrm{d}t}{t}.\end{aligned} \tag{2}$$

By Theorem 2.1, θ_a^F extends meromorphically to $\mathbb{C}$, and 0 is a regular value.

DEFINITION 1.2. – The Ray–Singer analytic torsion of X with coefficients in F is defined by $T_a(X,h^F) := \exp\{\frac{1}{2}\frac{\partial\theta_a^F}{\partial s}(0)\}$, and the Ray–Singer metric on the line $\det H^\bullet(X,F)$ is defined by

$$\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F)} := T_a\big(X,h^F\big)\,|\cdot|^{\mathrm{L}^2}_{\det H^\bullet(X,F)}. \tag{3}$$

2. Anomaly formulas for analytic torsion

The objects which follow will be defined more precisely in Section 3.

Using the notation in Section 3 to the metrics $g_s^{TX} = g^{TX}$, we denote by $\dot{R}^{TX}$, $\dot{R}^{TX}|Y$, $\dot{S}$ the corresponding forms defined in (11), (12). Then the following result generalizes [2, Theorem 7.10] to manifolds with boundary.

THEOREM 2.1. – *When $t\to 0$, for any $k\in\mathbb{N}$,*

$$\mathrm{Tr}_s\big[N\exp\big(-t^2D_a^2\big)\big] = \sum_{j=-1}^{k} a_j t^j + \mathcal{O}\big(t^{k+1}\big), \quad \textit{and} \tag{4}$$

$$\begin{aligned}a_{-1} &= \mathrm{rk}(F)\int_X\int^{B_X}\sum_{i=1}^m \frac{1}{2}e^i\wedge\widehat{e^i}\exp\bigg(-\frac{1}{2}\dot{R}^{TX}\bigg)\\ &\quad + \mathrm{rk}(F)\int_Y\int^{B_Y}\sum_{\alpha=1}^{m-1}\frac{1}{2}e^\alpha\wedge\widehat{e^\alpha}\sum_{k=0}^{\infty}\frac{\dot{S}^k}{2\Gamma\big(k/2+1\big)}\exp\bigg(-\frac{1}{2}\big(\dot{R}^{TX}|Y\big)\bigg),\\ a_0 &= \frac{m}{2}\chi(X,F).\end{aligned} \tag{5}$$

Let $\|\cdot\|_{\det F}$ be the metric on the line bundle $\det F$ induced by h^F.

Let (g_0^{TX},h_0^F) and (g_1^{TX},h_1^F) be two couples of metrics on TX and F. We will use the subscripts $0,1$ to distinguish the corresponding objects. Let ∇_j^{TX} $(j=0,1)$ be the Levi-Civita connection on (TX,g_j^{TX}) and put $\theta(F,h_1^F) := \mathrm{Tr}[(h_1^F)^{-1}\nabla^F h_1^F]$; this is a closed 1-form which vanishes if the metric $\|\cdot\|_{\det F,1}$ is flat, cf. [2, p. 63]. Let $E(TX,\nabla_0^{TX})$ be the relative Euler form of (TX,g_0^{TX}) defined by (17), let $\widetilde{E}(TX,\nabla_0^{TX},\nabla_1^{TX})$ be the secondary relative Euler class defined by (18), and let $B(\nabla_j^{TX})$ $(j=0,1)$ be the $m-1$-form on Y defined before (15). In (16), we define also the integral on X of a form in the relative complex $\Omega(X,Y,o(TX))$.

Now we can present our main result which generalizes [2, Theorem 0.1] to manifolds with boundary.

THEOREM 2.2. – *Let (g_0^{TX},h_0^F), (g_1^{TX},h_1^F) be two couples of metrics on TX and F. Then*

$$\log\bigg(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F),1}}{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F),0}}\bigg)^2 = (-1)^m\int_X\log\bigg(\frac{\|\cdot\|_{\det F,1}}{\|\cdot\|_{\det F,0}}\bigg)^2 E\big(TX,\nabla_0^{TX}\big)$$

$$+\int_X \widetilde{E}\left(TX, \nabla_0^{TX}, \nabla_1^{TX}\right)\theta\left(F, h_1^F\right) + \mathrm{rk}(F)\left[\int_Y B\left(\nabla_1^{TX}\right) - \int_Y B\left(\nabla_0^{TX}\right)\right]. \tag{6}$$

Outline of the proof. – We can guess the first two terms in the right-hand side of (6) by comparing with [2, Theorem 0.1], but the third term is more mysterious.

Let $s \in \mathbb{R} \to (g_s^{TX}, h_s^F)$ be a smooth family of metrics on TX, F. Let $*_s$ be the Hodge operator associated to the metrics g_s^{TX}. Let D_s be the operator D attached to the metrics (g_s^{TX}, h_s^F). Let $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F),s}$ be the corresponding Ray–Singer metric on $\det H^\bullet(X, F)$ and denote by $\exp(-tD_{s,a}^2)$ the heat semi-group of D_s^2 with boundary condition (1). Then as $t \to 0$, for any $k \in \mathbb{N}$, there is an asymptotic expansion

$$\mathrm{Tr}_s\left[\left(*_s^{-1}\frac{\partial *_s}{\partial s} + \left(h_s^F\right)^{-1}\frac{\partial h_s^F}{\partial s}\right)\exp\left(-tD_{s,a}^2\right)\right] = \sum_{j=-m}^{k} M_{j,s}t^{j/2} + \mathcal{O}\left(t^{k/2}\right). \tag{7}$$

Moreover,

$$\frac{\partial}{\partial s}\log\left(\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F),s}\right)^2 = M_{0,s}. \tag{8}$$

Eqs. (7), (8) generalize [2, Theorem 4.14] to manifolds with boundary, they generalize also [6, Theorem 3.27], [14, Theorem 7.3] to general metrics on F.

To prove Theorem 2.2, we need to calculate the asymptotic expansion of (7) when $t \to 0$. By using the local index technique in [2, §4(h)], we get the local contribution of (6) in the interior of X. To get the local contribution of (6) from the boundary, we use three ideas. First, we rescale the Clifford variables along Y, secondly, we use a special trivialization of the vector bundles involved adapted to the boundary situation, in order to get a manageable limiting boundary value problem (this special trivalization has already been used in [1, §13 (d), (e)], in a different context). Third, we introduce two extra Grassmann variables and a strange rescaling. □

3. Secondary classes for manifolds with boundary

In this section, we use the formalism of Berezin integrals to express certain characteristic classes which naturally arise in our anomaly formula (6).

For $\mathbb{Z}_2$-graded algebras $\mathcal{A}$, $\mathcal{B}$ with identity we introduce the $\mathbb{Z}_2$-graded tensor product $\mathcal{A}\widehat{\otimes}\mathcal{B}$ and define $\mathcal{A} := \mathcal{A}\widehat{\otimes} I$, and $\widehat{\mathcal{B}} := I\widehat{\otimes}\mathcal{B}$. Also, we write $\wedge := \widehat{\otimes}$. Let E and V be finite dimensional real vector spaces of dimension n and l, respectively. Assume that E is Euclidean and oriented, with oriented orthonormal basis $\{e_i\}_{i=1}^n$ and dual basis $\{e^i\}_{i=1}^n$. Then the Berezin integral [2, §3(a)], [11] is the linear map

$$\int^B : \Lambda V^* \wedge \widehat{\Lambda E^*} \to \Lambda V^*, \qquad \alpha \wedge \hat{\beta} \mapsto c_B\alpha\beta(e_1, \ldots, e_n), \tag{9}$$

where the normalizing constant is given by $c_B := (-1)^{n(n+1)/2}\pi^{-n/2}$. More generally, for any Euclidean vector space E with orientation line $o(E)$, the Berezin integral maps $\Lambda V^* \wedge \widehat{\Lambda E^*}$ into $\Lambda V^* \otimes o(E)$.

We now consider a smooth family of metrics, $\{g_s^{TX}\}_{s\in\mathbb{R}}$, on TX. We denote by g_s^{TY} the induced metrics on TY and by ∇_s^{TX}, ∇_s^{TY} the Levi-Civita connections on (TX, g_s^{TX}), (TY, g_s^{TY}), with curvatures R_s^{TX} and R_s^{TY}. Introduce the deformation space $X \times \mathbb{R}$, with projections $\pi_{\mathbb{R}} : X \times \mathbb{R} \to \mathbb{R}$ and $\pi_X : X \times \mathbb{R} \to X$, and canonical embedding $j : Y \times \mathbb{R} \hookrightarrow X \times \mathbb{R}$. The vertical bundle of the fibration $\pi_{\mathbb{R}}$ (resp. $Y \times \mathbb{R} \to \mathbb{R}$) will be denoted by $T\mathcal{X}$ (resp. $T\mathcal{Y}$); clearly, $T\mathcal{X} = \pi_X^* TX$ and $T\mathcal{Y} = (\pi_X \circ j)^* TY$. The bundle $T\mathcal{X}$ is naturally equipped with a metric, $g^{T\mathcal{X}}$, which coincides with g_s^{TX} over $X \times \{s\}$. Moreover, following [2, (4.50), (4.51)], there is a natural metric connection $\nabla^{T\mathcal{X}}$ on $T\mathcal{X}$ defined by

$$\nabla^{T\mathcal{X}} = \pi_X^*\nabla_s^{TX} + \mathrm{d}s \wedge \left(\mathcal{L}_{\partial/\partial s} + \frac{1}{2}\left(g_s^{TX}\right)^{-1}\mathcal{L}_{\partial/\partial s}g_s^{TX}\right) \tag{10}$$

with curvature $R^{T\mathcal{X}}$. We also denote by $\nabla^{T\mathcal{Y}}$ the connection on $T\mathcal{Y}$ defined as in (10) by g_s^{TY}, and denote $R^{T\mathcal{Y}}$ its curvature. Following [2, §3(e)], we view $R^{T\mathcal{X}}$ as a section of $\Lambda(T^*(X\times\mathbb{R}))\wedge\widehat{\Lambda(T^*\mathcal{X})}$ and write, with $\{e_i\}_{i=1}^m$ an orthonormal basis of $(T\mathcal{X},g^{T\mathcal{X}})$ and $\{e^i\}_{i=1}^m$ the corresponding dual basis of $T^*\mathcal{X}$,

$$\dot{R}^{T\mathcal{X}}=\frac{1}{2}\sum\nolimits_{1\leqslant k,l\leqslant m}\langle e_k,R^{T\mathcal{X}}e_l\rangle\widehat{e^k}\wedge\widehat{e^l}. \tag{11}$$

Near the boundary, we only consider orthonormal frames with the property that $e_m(y,s)=e_{\mathfrak{n}}$ is the inward pointing unit normal vector at $y\in Y$ with respect to the metric g_s^{TX}. Now let $\{e_\alpha\}_{1\leqslant\alpha\leqslant m-1}$ be a local orthonormal frame for $T\mathcal{Y}$, such that $\{e_\alpha\}_{1\leqslant\alpha\leqslant m-1}\cup\{e_m\}$ is an orthonormal frame for $T\mathcal{X}|(Y\times\mathbb{R})$. We set on $Y\times\mathbb{R}$,

$$\begin{aligned}&\dot{\mathcal{S}}=\frac{1}{2}\nabla^{T\mathcal{X}}\widehat{e^m}=\frac{1}{2}\sum_{1\leqslant\alpha,\beta\leqslant m-1}\langle\nabla^{T\mathcal{X}}_{e_\alpha}e_m,e_\beta\rangle e^\alpha\wedge\widehat{e^\beta},\\&\dot{R}^{T\mathcal{X}}|Y=\frac{1}{2}\sum_{1\leqslant\alpha,\beta\leqslant m-1}\langle e_\alpha,j^*R^{T\mathcal{X}}e_\beta\rangle\widehat{e^\alpha}\wedge\widehat{e^\beta},\qquad \dot{R}^{T\mathcal{Y}}=\frac{1}{2}\sum_{1\leqslant\gamma,\delta\leqslant m-1}\langle e_\gamma,R^{T\mathcal{Y}}e_\delta\rangle\widehat{e^\gamma}\wedge\widehat{e^\delta}.\end{aligned} \tag{12}$$

We will denote by $\int^{B_X}$, $\int^{B_Y}$ the Berezin integrals acting on $\Lambda(T^*X)\wedge\widehat{\Lambda(T^*X)}$, $\Lambda(T^*Y)\wedge\widehat{\Lambda(T^*Y)}$. We now put

$$e(T\mathcal{X},\nabla^{T\mathcal{X}})=\int^{B_X}\exp\Bigl(-\frac{1}{2}\dot{R}^{T\mathcal{X}}\Bigr),\qquad e(T\mathcal{Y},\nabla^{T\mathcal{Y}})=\int^{B_Y}\exp\Bigl(-\frac{1}{2}\dot{R}^{T\mathcal{Y}}\Bigr). \tag{13}$$

Then $e(T\mathcal{X},\nabla^{T\mathcal{X}})$ is an $o(TX)$-valued closed m-form on $X\times\mathbb{R}$, and $e(T\mathcal{Y},\nabla^{T\mathcal{Y}})$ is a closed $m-1$-form on $Y\times\mathbb{R}$ with values in the the orientation line bundle $o(TY)$ of TY.

Let $\psi(T\mathcal{X},\nabla^{T\mathcal{X}})$ be the Mathai–Quillen current on $T\mathcal{X}$ defined in [2, Def. 3.6] (*cf.* [11]). On $Y\times\mathbb{R}$, set

$$\begin{aligned}&e_b(Y\times\mathbb{R},\nabla^{T\mathcal{X}}):=e_m^*\psi(T\mathcal{X},\nabla^{T\mathcal{X}}),\\&B(\nabla^{T\mathcal{X}}):=\int_0^1\frac{\mathrm{d}u}{u}\int^{B_Y}\exp\Bigl(-\frac{1}{2}\bigl(\dot{R}^{T\mathcal{X}}|Y\bigr)+\bigl(1-u^2\bigr)\dot{\mathcal{S}}^2\Bigr)\sum_{k=1}^{\infty}\frac{(u\dot{\mathcal{S}})^k}{2\Gamma(k/2+1)}.\end{aligned} \tag{14}$$

The form $e_b(Y\times\mathbb{R},\nabla^{T\mathcal{X}})$ is an $o(TY)$-valued $m-1$-form on $Y\times\mathbb{R}$. If m is odd, then $e(T\mathcal{X},\nabla^{T\mathcal{X}})=0$ and, since $\dot{R}^{T\mathcal{Y}}=\dot{R}^{T\mathcal{X}}|Y-2\dot{\mathcal{S}}^2$, we obtain $e_b(Y\times\mathbb{R},\nabla^{T\mathcal{X}})=\frac{1}{2}e(T\mathcal{Y},\nabla^{T\mathcal{Y}})$.

For $j=0,1$, denote by $e(TX,\nabla_j^{TX})$ (resp. $e(TY,\nabla_j^{TY})$, $e_b(Y,\nabla_j^{TX})$, $B(\nabla_j^{TX})$) the restrictions of $e(T\mathcal{X},\nabla^{T\mathcal{X}})$ (resp. $e(T\mathcal{Y},\nabla^{T\mathcal{Y}})$, $e_b(Y\times\mathbb{R},\nabla^{T\mathcal{X}})$, $B(\nabla^{T\mathcal{X}})$) to $X\times\{j\}$ (resp. $Y\times\{j\}$), obviously, they depend only on the metric g_j^{TX}. If $Y=\emptyset$, $e(TX,\nabla_0^{TX})$ represents the Euler class $e(TX)$ of (TX,g_0^{TX}) in Chern–Weil theory. Hence $\chi(X,F):=\sum_{p=0}^m(-1)^p\dim H^p(X,F)$, the Euler characteristic of X with coefficients in F, is given in the case $Y=\emptyset$ by the Gauss–Bonnet–Chern theorem [7], $\chi(X,F)=\mathrm{rk}(F)\int_X e(TX)$. If $Y\neq\emptyset$, then by the Gauss–Bonnet–Chern theorem [7],

$$\chi(X,F)=\mathrm{rk}(F)\int_X e\bigl(TX,\nabla_0^{TX}\bigr)+(-1)^{m-1}\mathrm{rk}(F)\int_Y e_b\bigl(Y,\nabla_0^{TX}\bigr). \tag{15}$$

In [5], we give a new derivation of (15) by establishing the corresponding local index theorem by using heat kernel methods.

Put $\Omega^p(X,Y,o(TX))=\Omega^p(X,o(TX))\oplus\Omega^{p-1}(Y,o(TX))$ and define, for $(\sigma_1,\sigma_2)\in\Omega^p(X,Y,o(TX))$, $d(\sigma_1,\sigma_2)=(d\sigma_1,j^*\sigma_1-d\sigma_2)$, where we still denote by $j:Y\hookrightarrow X$ the canonical embedding. Then the complex $(\Omega(X,Y,o(TX)),d)$ calculates the relative cohomology $H^\bullet(X,Y,o(TX))$, cf. [4, p. 78]. For $(\sigma_1,\sigma_2)\in\Omega(X,Y,o(TX))$, $\sigma_3\in\Omega(X)$, we denote, cf. [4, p. 86],

$$\int_X (\sigma_1, \sigma_2) \wedge \sigma_3 := \int_X \sigma_1 \wedge \sigma_3 - \int_Y \sigma_2 \wedge \sigma_3 \tag{16}$$

this induces the Poincaré duality $H^\bullet(X, Y, o(TX)) \times H^\bullet(X, \mathbb{R}) \to \mathbb{R}$. For $j = 0, 1$,

$$E\big(TX, \nabla_j^{TX}\big) := \big(e\big(TX, \nabla_j^{TX}\big), e_b\big(Y, \nabla_j^{TX}\big)\big) \tag{17}$$

is closed in $\Omega(X, Y, o(TX))$ and defines the relative Euler class of TX, i.e. $E(TX, \nabla_j^{TX}) \in H^\bullet(X, Y, o(TX))$ does not depend on the choice of g_j^{TX}.

In the following, if β_1, β_2 are two forms on a manifold Z depending on $s \in \mathbb{R}$, then we write $[\beta_1 + ds \wedge \beta_2]^{ds} := \beta_2$.

We now define the secondary classes which appear in our final formula (6):

DEFINITION 3.1. – Set

$$\begin{aligned} &\widetilde{e}\big(TX, \nabla^{T\mathcal{X}}\big) = \int_0^1 ds\big[e\big(T\mathcal{X}, \nabla^{T\mathcal{X}}\big)\big]^{ds}, \qquad \widetilde{e_b}\big(TX, \nabla^{T\mathcal{X}}\big) = \int_0^1 ds\big[e_b\big(Y \times \mathbb{R}, \nabla^{T\mathcal{X}}\big)\big]^{ds}. \\ &\widetilde{E}\big(TX, \nabla_0^{TX}, \nabla_1^{TX}\big) = \big(\widetilde{e}\big(TX, \nabla^{T\mathcal{X}}\big), -\widetilde{e_b}\big(TX, \nabla^{T\mathcal{X}}\big)\big). \end{aligned} \tag{18}$$

If Y is empty, then $\widetilde{E}$ is the usual Chern–Simons class associated with the Euler class.

THEOREM 3.2. – *Modulo exact forms in the complex* $(\Omega(X, Y, o(TX)), d)$*,* $\widetilde{E}(TX, \nabla_0^{TX}, \nabla_1^{TX})$ *does not depend on the choice of the path* g_s^{TX} *from* g_0^{TX} *to* g_1^{TX}*. Moreover,*

$$d\widetilde{E}\big(TX, \nabla_0^{TX}, \nabla_1^{TX}\big) = E\big(TX, \nabla_1^{TX}\big) - E\big(TX, \nabla_0^{TX}\big). \tag{19}$$

It is obvious from Theorem 3.2 that $\widetilde{E}(TX, \nabla_0^{TX}, \nabla_1^{TX})$ defines the secondary relative Euler class of TX in the spirit of Chern–Simons theory.

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