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Non-parametric estimation from simultaneous degradation and failure time data

Vilijandas Bagdonavičius [a,b], Algis Bikelis [a], Vytautas Kazakevičius [a], Mikhail Nikulin [b]

[a] University of Vilnius, 24 Naugarduko, Vilnius, Lithuania

[b] Statistique Mathématique, Université Victor Segalen, 33076 Bordeaux, France

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Abstract The paper considers degradation and failure time models with multiple failure modes. Dependence of traumatic failure intensities on the degradation level are included into the models. Non-parametric estimators of various reliability characteristics are proposed. Theorems on simultaneous asymptotic distribution of random functions characterizing degradation and intensities of traumatic events are proposed. ***To cite this article: V. Bagdonavičius et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 183–188.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Estimation non-paramétrique à partir de données de dégradation et de défaillances

Résumé On considère l'inférence statistique pour des données qui répresentent des mesures de dégradation ainsi que des moments de défaillance (pannes) de plusieurs modalités. Les modèles tiennent compte de l'influence du niveau de dégradation sur les intensités des évenements traumatiques. Des estimateurs non-paramétriques de plusieurs caractéristiques de fiabilité sont proposés. Des théorèmes sur les propriétés asymptotiques des estimateurs sont proposés. ***Pour citer cet article : V. Bagdonavičius et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 183–188.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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On suppose qu'un processus de dégradation $Z(t)$ est donné par un modèle path (Meeker and Escobar [3])

$$Z(t) = g(t, A), \quad t \geqslant 0\,;$$

où $A = (A_1, \ldots, A_r)$ est un vecteur aléatoire de composantes positives et de fonction de répartition π, g est une fonction mesurable sur $\mathbf{R}_+^{r+1}$ et supposée connue, croissante et dérivable en t, $g(0, A) = 0$.

Notons h la fonction inverse de g par rapport au premier argument. Alors $T^{(0)} = h(z_0, A)$ est l'instant de la défaillance naturelle, i.e. l'instant quand la dégradation atteint le niveau donné z_0. Toute défaillance qui n'est pas naturelle est appellée traumatique. On suppose que des défaillances traumatiques de plusieurs modalités sont possibles. Notons $T^{(k)}$ $(k = 1, \ldots, s)$ le moment de défaillance qui correspond à la k-ième modalité de défaillance.

On suppose que les variables aléatoires $T^{(1)}, \ldots, T^{(s)}$ sont conditionnellement indépendantes (sachant que $A = a$) et leurs intensités ne dépendent que du niveau du processus de dégradation, i.e. la fonction de

E-mail address: Vilius@sm.u-bordeaux2.fr (V. Bagdonavičius).

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survie conditionnelle de $T^{(k)}$ est

$$S^{(k)}(t \mid a) = \exp\left(-\int_0^t \lambda^{(k)}\bigl(g(s,a)\bigr)\,\mathrm{d}s\right) = \exp\left(-\int_0^{g(t,a)} h'(z,a)\,\mathrm{d}\Lambda^{(k)}(z)\right), \tag{1}$$

où h' est la dérivé partielle de h par rapport au premier argument et

$$\Lambda^{(k)}(z) = \int_0^z \lambda^{(k)}(y)\,\mathrm{d}y. \tag{2}$$

On suppose que n unités sont testées et que les moments de défaillance

$$T_i = \min\bigl(T_i^{(0)}, \ldots, T_i^{(s)}\bigr),$$

les indicatrices des modalités de défaillances V_i et les trajectoires du processus de dégradation $g(t, A_i)$, $t \leqslant T_i$, sont observés ; ici $A_i = (A_{i1}, \ldots, A_{ir})$ et $V_i = k$, si $T_i = T_i^{(k)}$. Donc on a un modèle du type « risques compétitifs ».

Pour tout k $(k = 1, \ldots, s)$, on considère un processus de comptage

$$N^{(k)}(z) = \sum_{i=1}^n \mathbf{1}_{\{Z_i \leqslant z, V_i = k\}}, \quad z \in [0, z_0], \tag{3}$$

où $Z_i = g(T_i, A_i)$.

Le Théorème 1 donne l'expression du compensateur pour le processus $N^{(k)}(z)$. Ce théorème implique les estimateurs optimaux $\hat{\Lambda}^{(k)}(z)$ des intensités cumulatives.

Les Théorèmes 2 et 3 donnent les propriétés asymptotiques de l'estimateur $\hat{\Lambda}^{(k)}(z)$. Le Théorème 4 considère les propriétés asymptotiques de la loi conjointe des estimateurs $\hat{\Lambda}^{(k)}(z)$ et de l'estimateur optimal $\hat{\pi}$ de π. Les estimateurs asymptotiquement efficaces de plusieurs caractéristiques de fiabilité sont donnés. Le Théorème 5 donne les propriétés asymptotiques de ces estimateurs.

1. Introduction

Degradation data are obtained when not only failure times but also quantities characterizing degradation of units are measured.

We call a failure of a unit natural if the degradation attains a critical level. Other failures are called traumatic. Traumatic failures may be of various modes. The intensities of the traumatic failures of different modes depend on degradation.

Parametric estimation methods from simultaneous degradation and failure time data when the degradation process is gamma are given by Bagdonavičius and Nikulin [2]. We consider here non-parametric methods of estimation.

Suppose that the degradation process $Z(t)$ is modeled by the path model (see Meeker and Escobar [3])

$$Z(t) = g(t, A), \quad t \geqslant 0;$$

here $A = (A_1, \ldots, A_r)$ is a random vector of positive components with the distribution function π, and g is a specified measurable function, defined on $\mathbf{R}_+^{r+1}$, increasing and differentiable in t, $g(0, A) = 0$. For example, $g(t, A) = t/A$ (linear degradation, $r = 1$), $g(t, A) = (t/A_1)^{A_2}$ (convex or concave degradation, $r = 2$).

Denote by h the inverse function of g with respect to the first argument, by $T^{(0)} = h(z_0, A)$ the time of the natural failure, and by $T^{(k)}$ $(k = 1, \ldots, s)$ the failure time corresponding to the kth traumatic failure mode.

We suppose that the random variables $T^{(1)}, \ldots, T^{(s)}$ are conditionally independent (given $A = a$) and have the intensities depending only on the degradation level. It means that the conditional survival function of $T^{(k)}$ has the form (1), where h' is the partial derivative of h with respect to the first argument and $\Lambda^{(k)}(z)$ is given by (2).

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2. Non-parametric estimation of the cumulative intensities

Most reliability characteristics are functionals of the cumulative intensities $\Lambda^{(k)}$ and the distribution function π. So let us begin with the non-parametric estimation of these functions.

Suppose that n units are on test. The failure moments $T_i = \min(T_i^{(0)}, \ldots, T_i^{(s)})$, the indicators of the failure modes V_i and the paths of degradation processes $g(t, A_i)$, $t \leqslant T_i$, are observed; here $A_i = (A_{i1}, \ldots, A_{ir})$, $V_i = k$ if $T_i = T_i^{(k)}$. Note that any path of the stochastic process $g(t, A_i)$ is defined by the values of this path at any r points. Usually $r = 2$ (convex or concave degradation) or $r = 1$ (linear degradation) should be used.

Thus, the data are:

$$\big(T_1, g(t, A_1), t \leqslant T_1, V_1\big), \ldots, \big(T_n, g(t, A_n), t \leqslant T_n, V_n\big).$$

These data can also be defined as a collection of the following vectors:

$$(A_1, T_1, V_1), \ldots, (A_n, T_n, V_n).$$

Set $Z_i = g(T_i, A_i)$. It is the degradation value of the ith unit at the failure moment.

Let us consider estimation of the cumulative intensities $\Lambda^{(k)}(z)$. For any k $(k = 1, \ldots, s)$ define a counting process (3) on $[0, z_0]$. It is the number of units having a failure of the kth mode before or at the moment when the degradation attains the level z. Estimation of the cumulative intensities are based on these counting processes.

THEOREM 1. – *Let $\mathcal{F}_z$ be the σ-algebra generated by the random vectors $A_1, \ldots, A_n$ and $N_n^{(1)}(y), \ldots, N_n^{(s)}(y)$, $y \leqslant z$. Then the process $N_n^{(k)}(z)$ can be written as the sum*

$$N_n^{(k)}(z) = \int_0^z \lambda^{(k)}(y) Y(y)\,\mathrm{d}y + M_n^{(k)}(z),$$

where

$$Y(z) = \sum_{Z_i \geqslant z} h'(z, A_i),$$

and $M_n^{(k)}$ is a martingale with respect to the filtration $(\mathcal{F}_z,\ 0 \leqslant z \leqslant z_0)$. Moreover, the predictable covariation of the processes $M^{(k)}$ and $M_n^{(l)}$ is given by

$$\big\langle M^{(k)}, M^{(k)}\big\rangle(z) = \delta_{kl} \int_0^z \lambda^{(k)}(y) Y(y)\,\mathrm{d}y,$$

where $\delta_{kl} = \mathbf{1}_{\{k=l\}}$ stands for the Kronecker symbol.

The theorem implies that the optimal estimators of the cumulative intensities $\Lambda^{(k)}(z)$ are Nelson–Aalen type (*see* Andersen et al. [1], pp. 177–178):

$$\hat{\Lambda}^{(k)}(z) = \int_0^z Y^{-1}(y)\,\mathrm{d}N^{(k)}(y) = \sum_{Z_i \leqslant z,\ V_i = k} Y^{-1}(Z_i) = \sum_{Z_i \leqslant z,\ V_i = k} \left(\sum_{Z_j \geqslant Z_i} h'(Z_i, A_j)\right)^{-1}.$$

The estimator is correctly defined if $Z_i \leqslant z$ and $V_i = k$ for some i. If such i do not exist then the estimator is defined as 0.

Set

$$b(z) = \mathbf{E}\big\{h'(z, A)\mathbf{1}_{\{Z \geqslant z\}}\big\} = \int_{\mathbf{R}_+^r} h'(z, a) S\big(h(z, a) \mid a\big)\,\mathrm{d}\pi(a).$$

THEOREM 2. – *If $\sup_{0 \leqslant z \leqslant z_0} \mathbf{E}\{h'(z, A)\} < \infty$ then the estimator $\hat{\Lambda}^{(k)}$ is uniformly consistent, i.e. almost surely*

$$\sup_{0 \leqslant z \leqslant z_0} \big|\hat{\Lambda}^{(k)}(z) - \Lambda^{(k)}(z)\big| \to 0 \quad \text{as } n \to \infty.$$

Set $\Lambda = (\Lambda^{(1)}, \ldots, \Lambda^{(s)})$ and $\hat\Lambda = (\hat\Lambda^{(1)}, \ldots, \hat\Lambda^{(s)})$. Let $W = (W^{(1)}, \ldots, W^{(s)})$ be the vector of independent zero mean Gaussian processes with the covariance function

$$\mathbf{E}\big\{W^{(k)}(z)W^{(k)}(z')\big\} = \sigma_k^2(z \wedge z'),$$

where

$$\sigma_k^2(z) = \int_0^z \frac{\lambda^{(k)}(y)}{b(y)}\,\mathrm{d}y.$$

THEOREM 3. – *If* $\sup_{0\leqslant z\leqslant z_0} \mathbf{E}\{h'(z,A)\} < \infty$ *then the random vector function* $\sqrt{n}(\hat\Lambda - \Lambda)$ *tends to* W *weakly in the space* $D^s[0, z_0]$.

Set

$$\hat\pi(a) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{A_i \leqslant a\}} = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{A_{i1}\leqslant a_1,\ldots,A_{ir}\leqslant a_r\}}.$$

If the function π is continuous then the random function $\sqrt{n}(\hat\pi - \pi)$ tends in distribution in the Skorokhod space $D^r[0,\infty]$ to a zero mean Gaussian field $W^{(0)}$ with the covariance function

$$\mathbf{E}\big\{W^{(0)}(a)W^{(0)}(a')\big\} = \pi(a\wedge a') - \pi(a)\pi(a'),$$

where $a \wedge a' = (a_1 \wedge a'_1, \ldots, a_r \wedge a'_r)^{\mathrm{T}}$.

The following theorem is used for investigation of asymptotic properties of the reliability characteristics estimators.

THEOREM 4. – *If the function* π *is continuous and* $\sup_{0\leqslant z\leqslant z_0} \mathbf{E}\{h'(z,A)\} < \infty$ *then the random vector function*

$$\sqrt{n}(\hat\pi - \pi,\ \hat\Lambda - \Lambda)$$

tends weakly in the space $D^r[0,\infty] \times D^s[0,z_0]$ *to a random vector function with independent components.*

3. Non-parametric estimation of the main reliability characteristics

The failure time of a unit is the random variable

$$T = \min\big(T^{(0)}, T^{(1)}, \ldots, T^{(s)}\big).$$

Set

$$\hat\Lambda^{(\cdot)}(z) = \sum_{k=1}^s \hat\Lambda^{(k)}(z), \qquad \widehat{S}^{(k)}(t \mid a) = \exp\left(-\int_0^{g(t,a)} h'(z,a)\,\mathrm{d}\hat\Lambda^{(k)}(z)\right),$$

$$\widehat{S}^{(\cdot)}(t \mid a) = \exp\left(-\int_0^{g(t,a)} h'(z,a)\,\mathrm{d}\hat\Lambda^{(\cdot)}(z)\right).$$

The survival function and the mean of the random variable T are estimated by

$$\widehat{S}(t) = \int_{h(z_0,a)>t} \widehat{S}^{(\cdot)}(t \mid a)\,\mathrm{d}\hat\pi(a),$$

$$\hat e = \int_{\mathbf{R}_+^r} h(z_0,a)\,\mathrm{d}\hat\pi(a) - \int_{\mathbf{R}_+^r} \mathrm{d}\hat\pi(a)\int_0^{z_0} h'(z,a)\big\{h(z_0,a) - h(z,a)\big\}\widehat{S}^{(\cdot)}\big(h(z,a) \mid a\big)\,\mathrm{d}\hat\Lambda^{(\cdot)}(z).$$

The probability $P^{(k)}(t)$ of the traumatic failure of the kth mode in the interval $[0,t]$ is estimated by

$$\widehat{P}^{(k)}(t) = \int_{\mathbf{R}_+^r} \mathrm{d}\hat\pi(a)\int_0^{z_0\wedge g(t,a)} h'(z,a)\widehat{S}^{(\cdot)}\big(h(z,a) \mid a\big)\,\mathrm{d}\hat\Lambda^{(k)}(z),$$

the probability $P^{(tr)}(t)$ of a traumatic failure in the interval $[0,t]$ is estimated by

$$\widehat{P}^{(\mathrm{tr})}(t) = 1 - \int_{\mathbf{R}_+^r} \widehat{S}^{(\cdot)}\big(t \wedge h(z_0,a) \mid a\big)\,\mathrm{d}\hat{\pi}(a),$$

and the probability $P^{(0)}(t)$ of the natural failure in the interval $[0,t]$ is estimated by

$$\widehat{P}^{(0)}(t) = \int_{h(z_0,a)\leqslant t} \widehat{S}^{(\cdot)}\big(h(z_0,a) \mid a\big)\,\mathrm{d}\hat{\pi}(a).$$

Set

$$\sigma^2(z) = \sum_{k=1}^{s} \sigma_k^2(z), \qquad \sigma_{-k}^2(z) = \sigma^2(z) - \sigma_k^2(z).$$

THEOREM 5. – *Let g and $\hat{g}$ be any of above considered reliability characteristics and its estimator. If $\sup_{0\leqslant z\leqslant z_0} \mathbf{E}\{h'(z,A)\}^3 < \infty$ and the distribution function π is continuous then the estimator $\hat{g}$ is asymptotically efficient and the distribution of the random variable $\sqrt{n}(\hat{g}-g)$ tends to the normal law with the mean* 0 *and the variance $V(\hat{g})$, where*:

$$V\big(\widehat{S}(t)\big) = \int_0^{z_0} \left\{ \int_{y<g(t,a)\leqslant z_0} h'(y,a) S^{(\cdot)}(t \mid a)\,\mathrm{d}\pi(a) \right\}^2 \mathrm{d}\sigma^2(y) + \int_{g(t,a)<z_0} S^2(t \mid a)\,\mathrm{d}\pi(a) - S^2(t),$$

$$\begin{aligned} V(\hat{e}) = \int_0^{z_0} \Bigg\{ \int_{\mathbf{R}_+^r} h'(z,a) \bigg(\int_z^{z_0} h'(u,a) \big\{h(z,a) - h(u,a)\big\} S^{(\cdot)}\big(h(u,a) \mid a\big)\,\mathrm{d}\Lambda^{(\cdot)}(u) \\ - \big\{h(z_0,a) - h(z,a)\big\} S^{(\cdot)}\big(h(z_0,a) \mid a\big) \bigg) \mathrm{d}\pi(a) \Bigg\}^2 \mathrm{d}\sigma^2(z) \\ + \int_{\mathbf{R}_+^r} \bigg(h(z_0,a) - \int_0^{z_0} h'(z,a)\big\{h(z_0,a) - h(z,a)\big\} S^{(\cdot)}\big(h(z,a) \mid a\big)\,\mathrm{d}\Lambda^{(\cdot)}(z) \bigg)^2 \mathrm{d}\pi(a) - \mathrm{e}^2, \end{aligned}$$

$$\begin{aligned} V\big(\widehat{P}^{(k)}(t)\big) = \int_0^{z_0} \bigg(\int_{g(t,a)>z} h'(z,a) \int_z^{z_0\wedge g(t,a)} h'(u,a) S^{(\cdot)}\big(h(u,a) \mid a\big)\,\mathrm{d}\Lambda^{(k)}(u)\,\mathrm{d}\pi(a) \bigg)^2 \mathrm{d}\sigma_{-k}^2(z) \\ + \int_0^{z_0} \Bigg\{ \int_{g(t,a)>z} h'(z,a) \bigg(-\int_z^{z_0\wedge g(t,a)} h'(u,a) S^{(\cdot)}\big(h(u,a) \mid a\big)\,\mathrm{d}\Lambda^{(k)}(u) \\ + S^{(\cdot)}\big(h(z,a) \mid a\big) \bigg) \mathrm{d}\pi(a) \Bigg\}^2 \mathrm{d}\sigma_k^2(z) \\ + \int_{\mathbf{R}_+^r} \left\{ \int_0^{z_0\wedge g(t,a)} h'(z,a) S^{(\cdot)}\big(h(z,a) \mid a\big)\,\mathrm{d}\Lambda^{(k)}(z) \right\}^2 \mathrm{d}\pi(a) - \big\{P^{(k)}(t)\big\}^2, \end{aligned}$$

$$\begin{aligned} V\big(\widehat{P}^{(tr)}(t)\big) = \int_0^{z_0} \bigg(\int_{g(t,a)>z} h'(z,a) S^{(\cdot)}\big(t \wedge h(z_0,a) \mid a\big)\,\mathrm{d}\pi(a) \bigg)^2 \mathrm{d}\sigma^2(z) \\ + \int_{\mathbf{R}_+^r} \big\{S^{(\cdot)}\big(t \wedge h(z_0,a) \mid a\big)\big\}^2 \mathrm{d}\pi(a) - \left\{ \int_{\mathbf{R}_+^r} S^{(\cdot)}\big(t \wedge h(z_0,a) \mid a\big)\,\mathrm{d}\pi(a) \right\}^2, \end{aligned}$$

$$\begin{aligned} V\big(\widehat{P}^{(0)}(t)\big) = \sigma^2(z_0) \bigg(\int_{g(t,a)>z} h'(z,a) S^{(\cdot)}\big(h(z_0,a) \mid a\big)\,\mathrm{d}\pi(a) \bigg)^2 \\ + \int_{h(z_0,a)\leqslant t} \big\{S^{(\cdot)}\big(t \wedge h(z_0,a) \mid a\big)\big\}^2 \mathrm{d}\pi(a) - \big\{P^{(0)}(t)\big\}^2. \end{aligned}$$

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