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Extreme values of particular non-linear processes

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Abstract We investigate the asymptotic behavior of the maxima of a general class of deterministic chaotic processes – including the tent map and the logistic map –, of noisy chaotic processes, and of the Gaussian long memory k-factor Gegenbauer processes. ***To cite this article: D. Guégan, S. Ladoucette, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 73–78.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Valeurs extrêmes pour des processus non linéaires particuliers

Résumé Nous étudions le comportement asymptotique des maxima d'une classe générale de processus chaotiques déterministes – comprenant les applications tent et logistique –, de processus chaotiques bruités et des processus longue mémoire gaussiens de Gegenbauer à k facteurs. ***Pour citer cet article : D. Guégan, S. Ladoucette, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 73–78.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Dans cette Note, nous nous intéressons au comportement extremal de certains processus stationnaires non linéaires à partir du comportement asymptotique de leurs maxima. Nous considérons d'abord les processus générés par une famille paramétrique d'applications chaotiques définie par le système (3), où le paramètre ν est tel que $1/2 \leqslant \nu \leqslant 2$ (*voir* Hall et Wolff [9] et Lawrance et Spencer [11]). Nous considérons aussi les processus générés par une version bruitée du système chaotique (3) pris en $\nu = 2$ définie par le système (4), où $(\varepsilon_n)_{n\in\mathbb{N}}$ est une suite de variables aléatoires indépendantes et identiquement distribuées de densité $\beta(a, b)$ $(a, b > 0)$, indépendantes de $(X_n)_{n\in\mathbb{N}}$. Enfin, nous considérons les processus longue mémoire de Gegenbauer à k facteurs définis par l'équation (5), où les paramètres sont tels que $|d_i| < 1/2$ si $|\upsilon_i| < 1$, $|d_i| < 1/4$ si $|\upsilon_i| = 1$, $d_i \neq 0$, pour $i = 1, \ldots, k$, et où $(\varepsilon_n)_{n\in\mathbb{Z}}$ est un bruit blanc gaussien (*voir* Giraitis et Leipus [4] et Woodward et al. [14]).

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Dans la suite, $(M_n)_{n\in\mathbb{N}}$ désigne la suite du maximum $M_n = \max(Z_1,\ldots,Z_n)$, $n \geqslant 1$, associée à un processus $(Z_n)_{n\in\mathbb{N}}$ de fonction de distribution marginale F. Les Théorèmes 2, 3 et 4 établissent la fonction de distribution limite de la suite $(M_n)_{n\in\mathbb{N}}$ associée respectivement aux processus dépendants (3), (4) et (5).

THÉORÈME 2. – *Soit $(X_n)_{n\in\mathbb{N}}$ un processus stationnaire associé au système* (3) *avec ν fixé, dont la fonction de densité f est continue et strictement positive en* $1/2$. *Si $(X_n)_{n\in\mathbb{N}}$ possède un indice extrémal vérifiant* $0<\theta\leqslant 1$, *alors F appartient au domaine d'attraction de Weibull d'indice* $1/\nu$, *i.e.* :

$$\mathbb{P}\left\{c_n^{-1}(M_n-d_n)\leqslant x\right\}\to \Psi_{1/\nu}(x)$$

avec $c_n=(f(1/2)\theta n)^{-\nu}$ *et* $d_n=1$.

THÉORÈME 3. – *Soit* $(Y_n)_{n\in\mathbb{N}}$ *un processus stationnaire associé au système* (4) *avec* $(\varepsilon_n)_{n\in\mathbb{N}}$ *de densité* $\beta(a,b)$ $(a,b>0)$. *Si* $(Y_n)_{n\in\mathbb{N}}$ *possède un indice extrémal vérifiant* $0<\theta\leqslant 1$, *alors F appartient au domaine d'attraction de Weibull d'indice* $(2b+1)/2$, *i.e.* :

$$\mathbb{P}\left\{c_n^{-1}(M_n-d_n)\leqslant x\right\}\to \Psi_{(2b+1)/2}(x)$$

avec $c_n=\left(\frac{4\Gamma(a+b)\theta n}{(2b+1)\pi\Gamma(a)\Gamma(b)}\right)^{-2/(2b+1)}$ *et* $d_n=2$, *où* Γ *est la fonction Gamma.*

THÉORÈME 4. – *Soit* $(X_n)_{n\in\mathbb{Z}}$ *un processus stationnaire de Gegenbauer à k facteurs gaussien* (5) *d'écart-type σ, tel que* $\max(d_1,\ldots,d_k)>0$. *Alors, F appartient au domaine d'attraction de Gumbel, i.e.* :

$$\mathbb{P}\left\{c_n^{-1}(M_n-d_n)\leqslant x\right\}\to \Lambda(x)$$

avec $c_n=\sigma(2\log n)^{-1/2}$ *et* $d_n=\sigma\left((2\log n)^{1/2}-\frac{\log\log n+\log 4\pi}{2(2\log n)^{1/2}}\right)$.

Les Théorèmes 2 et 3 sont basés sur l'existence et la non nullité de l'indice extrémal θ défini en (2). Ne prouvant pas ce fait de manière théorique, nous proposons une méthode permettant l'estimation empirique de cet indice. Ainsi, toutes les constantes de normalisations apparaissant dans les différents théorèmes peuvent être estimées, et les résultats de cette Note peuvent être utilisés pour faire des calculs de risque par le biais des quantiles extrêmes.

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1. Introduction and background

Extreme value theory is an area of statistics devoted to the development of models and techniques for estimating the behavior of unusual or rare events which are important in several domains, such as meteorology, physics, biology, finance and internet traffic. In this Note, we investigate this theory for a family of chaotic maps (the generalized tent family) representing deterministic dynamical systems, with or without measurement noise, and for a particular class of long memory processes.

Throughout this Note, we denote by $(M_n)_{n\in\mathbb{N}}$ the sequence of the maximum defined by $M_n = \max(Z_1,\ldots,Z_n)$, $n\geqslant 1$, associated with a process $(Z_n)_{n\in\mathbb{N}}$ with marginal distribution function F, x_F the right end point of a distribution function F, and $g\in\mathcal{R}_\delta^{+\infty}$ a positive Lebesgue measurable function on $]0,+\infty[$ which is regularly varying at $+\infty$ with index $\delta\in\mathbb{R}$.

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) random variables. The classical extreme value theory deals with the distribution of M_n as $n\to+\infty$, and its central result is the following theorem derived by Fisher and Tippett [3] and first proved rigorously by Gnedenko [5]. This theorem gives the possible limiting forms for the distribution of M_n under linear normalizations.

THEOREM 1. – *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables. If for some sequences of constants $c_n > 0$ and $d_n \in \mathbb{R}$ we have*:

$$\mathbb{P}\left\{c_n^{-1}(M_n - d_n) \leqslant x\right\} \to H(x) \tag{1}$$

for some non-degenerate distribution function H, then H belongs to the type of one of the following three distribution functions:

$$\textit{Fréchet}: \quad \Phi_\alpha(x) = \begin{cases} 0, & x \leqslant 0, \\ \exp(-x^{-\alpha}), & x > 0,\ \alpha > 0, \end{cases}$$

$$\textit{Weibull}: \quad \Psi_\alpha(x) = \begin{cases} \exp(-(-x)^{\alpha}), & x \leqslant 0,\ \alpha > 0, \\ 1, & x > 0, \end{cases}$$

$$\textit{Gumbel}: \quad \Lambda(x) = \exp\left(-e^{-x}\right), \quad x \in \mathbb{R}.$$

We say that the distribution function F of a sequence of random variables $(Z_n)_{n\in\mathbb{N}}$ belongs to the domain of attraction of a distribution function H, and we write $F \in \mathcal{D}(H)$, if (1) holds for some sequences $c_n > 0$ and $d_n \in \mathbb{R}$. Necessary and sufficient conditions are known for each type of limit, involving the behavior of the tail $1 - F(x)$ (denoted by $\overline{F}(x)$) as x increases to infinity.

Before defining the three kinds of models that we consider in this Note, we recall the definition of the extremal index which turns out to be useful in the sequel. Following Leadbetter et al. [12], a stationary process $(Z_n)_{n\in\mathbb{N}}$ has an extremal index θ $(0 \leqslant \theta \leqslant 1)$ if for every $\tau > 0$ there exits a sequence $(u_n)_{n\in\mathbb{N}}$ such that:

$$\lim_{n\to+\infty} n\left(1 - F(u_n)\right) = \tau \quad \text{and} \quad \lim_{n\to+\infty} \mathbb{P}(M_n \leqslant u_n) = \mathrm{e}^{-\theta\tau}. \tag{2}$$

First of all, we consider the processes generated by a parametric family of chaotic maps defined by:

$$X_{n+1} = \varphi_\nu(X_n) = \begin{cases} 1 - (1 - 2X_n)^\nu, & 0 \leqslant X_n < 1/2, \\ 1 - (2X_n - 1)^\nu, & 1/2 \leqslant X_n \leqslant 1, \end{cases} \tag{3}$$

where $1/2 \leqslant \nu \leqslant 2$. These processes are used for instance in engineering systems, see Lawrance and Spencer [11]. This family of chaotic maps contains the tent map ($\nu = 1$) and the logistic map ($\nu = 2$) and it is referred to as the generalized tent family, see Hall and Wolff [9]. Invariant densities associated with the processes (3) are only known for $\nu = 1/2$ (the density is $f(x) = 2(1-x)$, $x \in [0,1]$), $\nu = 1$ (the density is uniform on $[0,1]$) and $\nu = 2$ (the $\beta(1/2, 1/2)$ density).

In the sequel, we also consider the processes generated by the following scheme:

$$\begin{cases} Y_n = X_n + \varepsilon_n, & n \in \mathbb{N}, \\ X_n = 4X_{n-1}(1 - X_{n-1}), & n \in \mathbb{N}^*, \end{cases} \tag{4}$$

which is a noisy version of the system (3) when $\nu = 2$, where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables independent of $(X_n)_{n\in\mathbb{N}}$ with a density which belongs to the $\beta(a,b)$ class $(a, b > 0)$. Such processes with an additive noise whose density has a finite support are useful to take into account measurement errors. They are extensively used in physics, see for instance Loreto et al. [13]. We choose this class of densities for the noise since it contains the uniform density ($a = b = 1$) and the $\beta(1/2, 1/2)$ density.

The last model we consider is the Gaussian long memory k-factor Gegenbauer process $(X_n)_{n\in\mathbb{Z}}$ defined by:

$$\prod_{i=1}^{k}\left(I - 2\upsilon_i B + B^2\right)^{d_i} X_n = \varepsilon_n \tag{5}$$

with $|d_i| < 1/2$ if $|\upsilon_i| < 1$, $|d_i| < 1/4$ if $|\upsilon_i| = 1$, $d_i \neq 0$ for $i = 1, \ldots, k$, and $(\varepsilon_n)_{n\in\mathbb{Z}}$ a Gaussian white noise (*see* Giraitis and Leipus [4] and Woodward et al. [14]). When $\max(d_1, \ldots, d_k) > 0$, the process (5) is persistent since its spectral density explodes at least at one frequency. We investigate this process because recently it has been proved that specific chaotic systems can also present long memory behavior, see Guégan and Ladoucette [7] for the general logistic map and Guégan [6] for systems in higher dimension.

In the following section, we provide some results related to the extremal behavior of the processes (3), (4) and (5) through the asymptotic behavior of their maxima.

2. Maxima's limiting distribution of processes (3), (4) and (5)

The set of the invariant densities associated with the processes (3) which are continuous and strictly positive at $1/2$ is non empty (*see* the cases $\nu = 1/2$, 1 and 2). Taking this property into account, we provide in Theorem 2 the domain of attraction of the marginal distribution function associated with the process (3).

THEOREM 2. – *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary process associated with the system* (3) *for a fixed ν whose density function f is continuous and strictly positive at* $1/2$. *If $(X_n)_{n\in\mathbb{N}}$ has an extremal index $0 < \theta \leqslant 1$, then the distribution function F of the process belongs to the domain of attraction of the Weibull distribution with index $1/\nu$, i.e.*:

$$\mathbb{P}\left\{c_n^{-1}(M_n - d_n) \leqslant x\right\} \to \Psi_{1/\nu}(x)$$

with $c_n = (f(1/2)\theta n)^{-\nu}$ and $d_n = 1$.

For instance, we get $F \in \mathcal{D}(\Psi_1)$, $c_n = (\theta n)^{-1}$ and $d_n = 1$ for the tent process ($\nu = 1$), and $F \in \mathcal{D}(\Psi_{1/2})$, $c_n = (\pi/(2\theta n))^2$ and $d_n = 1$ for the logistic process ($\nu = 2$).

In Theorem 2 we assume that the extremal index θ of the process (3) exists and is different from zero. This assumption is also made in the next theorem for the process (4). Since we do not provide the theoretical value of θ in these two theorems, we shall discuss a method for estimating it in the next section.

In the following theorem, we provide the domain of attraction of the marginal distribution function of the process (4).

THEOREM 3. – *Let $(Y_n)_{n\in\mathbb{N}}$ be a stationary process associated with the system* (4), *assuming $(\varepsilon_n)_{n\in\mathbb{N}}$ has a $\beta(a,b)$ density $(a, b > 0)$. If $(Y_n)_{n\in\mathbb{N}}$ has an extremal index $0 < \theta \leqslant 1$, then the distribution function F of the process belongs to the domain of attraction of the Weibull distribution with index $(2b+1)/2$, i.e.*:

$$\mathbb{P}\left\{c_n^{-1}(M_n - d_n) \leqslant x\right\} \to \Psi_{(2b+1)/2}(x)$$

with $c_n = \left(\frac{4\Gamma(a+b)\theta n}{(2b+1)\pi\Gamma(a)\Gamma(b)}\right)^{-2/(2b+1)}$ and $d_n = 2$, where Γ denotes the Gamma function.

We now establish the limiting distribution of the suitably normalized maximum associated with the Gaussian long memory process (5) when it is persistent.

THEOREM 4. – *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary Gaussian k-factor Gegenbauer process* (5) *with standard deviation σ, such that* $\max(d_1, \ldots, d_k) > 0$. *Then, the distribution function F of the process belongs to the domain of attraction of the Gumbel distribution, i.e.*:

$$\mathbb{P}\left\{c_n^{-1}(M_n - d_n) \leqslant x\right\} \to \Lambda(x)$$

with $c_n = \sigma(2\log n)^{-1/2}$ and $d_n = \sigma\left((2\log n)^{1/2} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{1/2}}\right)$.

In establishing the limiting distribution of their maxima, we extend the Theorem 1 of Fisher–Tippett [3] to the non-linear dependent processes (3), (4) and (5). Proofs are given in Section 4 and details can be found in Guégan and Ladoucette [8].

3. Estimation of the extremal index

Theorems 2 and 3 are based on the assumption of the existence of the extremal index θ and of its strict positivity. In this section, we propose to estimate θ for both processes (3) and (4) using simulations.

We choose the blocks method which involves dividing N given data into m blocks of length n and setting a high threshold u. A natural estimator of θ is $\hat{\theta} = K_u/N_u$, where K_u and N_u are respectively the number of blocks and the number of observations that exceed the threshold. In order to get the properties of this estimator, we propose a simulation approach. Indeed, Hsing [10] has proved the asymptotic normality of this estimator under particular conditions, but we are unable to verify these conditions for the processes we consider here. The general method that we use is the following. We simulate s realizations of length $N = mn$ of a process with s different initial conditions X_0. We obtain s estimated values $\hat{\theta}_1, \ldots, \hat{\theta}_s$ of θ and we calculate $\hat{\theta} = (\sum_{i=1}^{s} \hat{\theta}_i)/s$ and its standard deviation $\hat{\sigma}_\theta = \sqrt{(\sum_{i=1}^{s}(\hat{\theta}_i - \hat{\theta})^2)/s}$.

As examples, we consider the system (3) with $\nu = 1$ and $\nu = 2$. In both cases, we fix $m = n = 200$ and $s = 100$. The threshold u is chosen such that the ratios K_u/m and N_u/mn are small enough. Then, for $\nu = 1$ we choose $u = 0.9950$ and for $\nu = 2$ we choose $u = 0.9999$. We obtain $\hat{\theta} = 0.98$ and $\hat{\sigma}_{\hat{\theta}} = 0.02$ for the tent process ($\nu = 1$), and $\hat{\theta} = 0.99$ and $\hat{\sigma}_{\hat{\theta}} = 0.01$ for the logistic process ($\nu = 2$). Other examples can be found in Guégan and Ladoucette [8].

The extremal index is a measure of the clustering tendency of extremes: $\theta = 1$ for sequences of i.i.d. random variables but the converse is false. Here, it turns out that the estimated values of θ are close to one whereas the simulated processes are dependent.

Using the estimate obtained for θ, we can deduce $\hat{c}_n$, an estimate for the normalizing constants c_n which appear in both Theorems 2 and 3. Then, we can provide estimates for the normalizing constants which appear in all the theorems and the results of this Note can be used for computing quantile risk measures.

4. Proofs

Proof of Theorem 2. – We assume that $(X_n)_{n\in\mathbb{N}}$ has an extremal index $0 < \theta \leqslant 1$. We have $x_F = 1$. Following Hall and Wolff [9], we consider the pre-image of a general point $1 - x^{-1},\ x > 1$, and we get:

$$f\left(1 - x^{-1}\right) = (2\nu)^{-1} x^{1-1/\nu}\left(f\left(1/2\left(1 + x^{-1/\nu}\right)\right) + f\left(1/2\left(1 - x^{-1/\nu}\right)\right)\right).$$

Then, we have the asymptotic expansion $f(1 - x^{-1}) \sim \nu^{-1} f(1/2) x^{1-1/\nu}$ as $x \to +\infty$, i.e. $f(1 - x^{-1}) \in \mathcal{R}_{1-1/\nu}^{+\infty}$. Using a Karamata's theorem (*see* Theorem 1, p. 281 in Feller [2]), we have $\overline{F}(1 - x^{-1}) \sim f(1/2) x^{-1/\nu}$ as $x \to +\infty$, i.e. $\overline{F}(1 - x^{-1}) \in \mathcal{R}_{-1/\nu}^{+\infty}$.

Using Theorem 1.6.2 and Corollary 3.7.3 of Leadbetter et al. [12], we conclude that $F \in \mathcal{D}(\Psi_{1/\nu})$, $c_n = (f(1/2)\theta n)^{-\nu}$ and $d_n = 1$, and the theorem is proved.

Proof of Theorem 3. – Assume that $(Y_n)_{n\in\mathbb{N}}$ has an extremal index $0 < \theta \leqslant 1$. The invariant densities of $(Y_n)_{n\in\mathbb{N}}$, $(X_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ are respectively denoted by f, g and h. It is obvious that $x_F = 2$. For $x \geqslant 1$, we have:

$$\begin{aligned} f\left(2 - x^{-1}\right) &= \int_{1-x^{-1}}^{1} g(y) h\left(2 - x^{-1} - y\right) \mathrm{d}y \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_{x}^{+\infty} y^{-2} g\left(1 - y^{-1}\right)\left(x^{-1} - y^{-1}\right)^{b-1}\left(1 + y^{-1} - x^{-1}\right)^{a-1} \mathrm{d}y. \end{aligned}$$

Using Karamata's theorem, we get $f(2 - x^{-1}) \sim \frac{2\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{-b} g(1 - x^{-1})$ as $x \to +\infty$, i.e. $f(2 - x^{-1}) \in \mathcal{R}_{(1-2b)/2}^{+\infty}$. Then, we apply again Karamata's theorem to obtain

$$\overline{F}\left(2-x^{-1}\right)\sim\frac{2}{2b+1}x^{-1}f\left(2-x^{-1}\right)\sim\frac{4\Gamma(a+b)}{(2b+1)\Gamma(a)\Gamma(b)}x^{-(b+1)}g\left(1-x^{-1}\right)$$
$$\sim\frac{4\Gamma(a+b)}{(2b+1)\pi\Gamma(a)\Gamma(b)}x^{-(2b+1)/2}$$

as $x\to+\infty$, i.e. $\overline{F}(2-x^{-1})\in\mathcal{R}^{+\infty}_{-(2b+1)/2}$.

Using Theorem 1.6.2 and Corollary 3.7.3 of Leadbetter et al. [12], we conclude that $F\in\mathcal{D}(\Psi_{(2b+1)/2})$, $c_n=\left(\frac{4\Gamma(a+b)\theta n}{(2b+1)\pi\Gamma(a)\Gamma(b)}\right)^{-2/(2b+1)}$ and $d_n=2$, and the theorem is proved.

Proof of Theorem 4. – A stationary Gaussian process whose autocovariance function γ satisfies the decay rate condition $\gamma(n)\log n\to 0$ as $n\to+\infty$ of Berman [1] belongs to the Gumbel domain with the same normalizing constants as in the i.i.d. case (*see* Theorem 4.3.3 of Leadbetter et al. [12]). Since it has a causal representation, the process (5) is Gaussian. From the expression of the asymptotic behavior of its autocovariance function γ given in Giraitis and Leipus [4], we obtain:

$$\left|\gamma(n)\right|\leqslant\sum_{i=1,\dots,k:d_i>0}An^{2d_i^*-1}\left(B+\mathrm{o}(1)\right)\quad\text{as } n\to+\infty$$

with $d_i^*=d_i$ if $|\upsilon_i|<1$ and $d_i^*=2d_i$ if $|\upsilon_i|=1$ ($i=1,\dots,k:d_i>0$), and with A and B two finite positive constants. Since $2d_i^*-1<0$, we have $n^{2d_i^*-1}\log n\to 0$ as $n\to+\infty$. Hence, Berman's condition holds, and the theorem is proved.

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