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On the invariance of the semigroup of a quasi-ordinary surface singularity

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Abstract We give an algebraic proof for 2-dimensional germs of the analytic invariance of a semigroup associated by González Pérez to any irreducible germ $\mathcal{S}$ of complex quasi-ordinary hypersurface. We deduce from it a new proof of the analytic invariance of the normalized characteristic exponents. Moreover, we associate values in the semigroup to the elements of a subset of the local algebra of $\mathcal{S}$. ***To cite this article: P. Popescu-Pampu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1101–1106.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur l'invariance du semi-groupe d'une singularité quasi-ordinaire de surface

Résumé Nous donnons une preuve algébrique dans le cas des germes bidimensionnels de l'invariance analytique d'un semi-groupe associé par González Pérez à tout germe quasi-ordinaire irréductible $\mathcal{S}$ d'hypersurface complexe. Nous en déduisons une nouvelle preuve de l'invariance analytique des exposants caractéristiques normalisés. De plus, nous associons des valeurs dans le semi-groupe aux éléments d'un sous-ensemble de l'algèbre locale de $\mathcal{S}$. ***Pour citer cet article : P. Popescu-Pampu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1101–1106.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Cette note expose de manière abrégée une partie des résultats de [14]. L'objet principal d'étude est celui de germe *quasi-ordinaire* $(\mathcal{S}, 0)$ d'espace analytique complexe réduit (Définition 2.1). Ces germes apparaissent naturellement dans la méthode de Jung de résolution des singularités d'un germe équidimensionnel d'espace par résolution plongée du lieu discriminant d'un morphisme fini vers un espace lisse de même dimension (*voir* [7] pour la dimension 2).

Dans [5,6], González Pérez a introduit un semi-groupe $\Gamma(f)$ associé à un polynôme irréductible f qui définit un germe quasi-ordinaire $\mathcal{S}$. En utilisant l'invariance topologique des exposants caractéristiques normalisés obtenue par Gau et Lipman [4,10], il a prouvé que, à isomorphisme de semi-groupes près, $\Gamma(f)$ ne dépend pas du choix de f et qu'il est un invariant complet du type topologique plongé de $\mathcal{S}$, donc a fortiori un invariant de son type analytique.

Nous donnons ici une preuve algébrique de l'invariance analytique de ce semi-groupe dans le cas où $\mathcal{S}$ est de dimension 2 (Corollaire 3.4), en prouvant que le semi-groupe construit par González Pérez est isomorphe

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à un semi-groupe que nous construisons uniquement à partir de $\mathcal{S}$, ne dépendant d'aucun morphisme quasi-ordinaire (Théorème 3.3). Décrivons brièvement notre construction.

Nous supposerons dans ce qui suit que $\mathcal{S}$ n'est pas lisse en 0. Soit $\nu : \overline{\mathcal{S}} \to \mathcal{S}$ le morphisme de normalisation du germe $(\mathcal{S}, 0)$ et $\mathcal{A}$ son algèbre locale. Soit $\mu : \mathcal{R} \to \overline{\mathcal{S}}$ la résolution minimale des singularités de $\overline{\mathcal{S}}$. Définissons le morphisme $\eta : \overline{\mathcal{R}} \to \mathcal{R}$ comme étant l'éclatement de la préimage de 0 si le lieu réduit $(\nu \circ \mu)^{-1}(\mathrm{Sing}(\mathcal{S}))$ est lisse et l'identité sur $\mathcal{R}$ sinon. Posons enfin $\theta := \nu \circ \mu \circ \eta$.

Nous disons que le germe $\mathcal{S}$ est *quadratique ordinaire* si l'algèbre $\mathcal{A}$ est isomorphe à $\mathbf{C}\{X_1, X_2, Y\}/(Y^2 - X_1X_2)$.

LEMME 3.1. – *Si le germe $\mathcal{S}$ n'est pas quadratique ordinaire, le sous-espace réduit* $\theta^{-1}(\mathrm{Sing}(\mathcal{S}))$ *de la surface lisse $\overline{\mathcal{R}}$ est un diviseur à croisements normaux qui admet au moins un point singulier.*

Supposons dorénavant que $\mathcal{S}$ n'est pas quadratique ordinaire et que P est un point singulier du diviseur réduit $\theta^{-1}(\mathrm{Sing}(\mathcal{S}))$. Considérons des coordonnées locales $X^P := (X_1^P, X_2^P)$ en P qui définissent le lieu $\theta^{-1}(\mathrm{Sing}(\mathcal{S}))_P$. Alors le sous-ensemble suivant de l'algèbre $\mathcal{A}$ ne dépend pas du choix de X^P en P :

$$\mathcal{D}_P(\mathcal{S}) := \left\{h \in \mathcal{A} - \{0\},\ \theta^*(h)_P \text{ a un terme dominant dans les coordonnées } X^P\right\}.$$

DÉFINITION 3.2. – Le *semi-groupe de $\mathcal{S}$ associé à P*, noté $\Gamma_P(\mathcal{S})$, est le sous-semi-groupe suivant de $(\mathbf{N}^2, +) : \Gamma_P(\mathcal{S}) := \{v_{X^P}(\theta^*(h)_P),\ h \in \mathcal{D}_P(\mathcal{S})\}$.

En abrégé, notre résultat principal est le suivant :

THÉORÈME 3.1. – *Le semi-groupe $\Gamma(f)$ est naturellement isomorphe au semi-groupe $\Gamma_P(\mathcal{S})$.*

Dans la Section 1 nous expliquons nos motivations, puis dans la Section 2 nous donnons les définitions principales de ce travail. Dans la Section 3 nous énonçons nos résultats et dans la Section 5 nous esquissons la preuve du Théorème 3.1. Auparavant, nous dédions la Section 4 à l'un des outils principaux de cette preuve, les développements suivant les *semi-racines* d'un polynôme quasi-ordinaire.

1. Motivations

A classical way to study a germ C of irreducible complex analytic plane curve is to introduce its Newton–Puiseux series in some coordinate system (X, Y), which allows one to define its so-called characteristic exponents. If the coordinates are generic – which means that the Y-axis and the tangent cone to the curve are transversal – the characteristic exponents do not depend of them and their collection is a complete invariant of the embedded topological type of the curve.

Another way to study the germ is to associate to it a semigroup Γ, the set of intersection numbers of C with other germs. If a germ D is defined by a function g, the intersection number (C, D) is equal to the order of vanishing of $\nu^*(g\,|_C)$ at the base point of $\overline{C}$, where $\nu : \overline{C} \to C$ is the normalisation morphism of C. This shows that Γ depends only of the analytic type of the germ C.

An isomorphic semigroup is obtained considering the orders of the series $g(\xi)$, where g varies in $\mathbf{C}\{X, Y\}$ and ξ is a Newton–Puiseux series of C in the coordinates (X, Y). Seen as an abstract semigroup, Γ is also a complete invariant of the embedded topological type of C. For the preceding claims, see [15] and [13].

In [5,6], González Pérez introduced an analogous semigroup associated to an irreducible quasi-ordinary hypersurface germ by using a notion of initial form in a ring graded essentially by Newton polyhedra. Using the topological invariance of the normalized characteristic exponents, proved by Gau and Lipman [4,10], he showed that up to isomorphism, this semigroup does not depend on the quasi-ordinary projection. Moreover, it is a complete invariant of the embedded topological type of the germ, and a fortiori it is an analytic invariant of the germ.

In the case of surfaces, we give an algebraic proof of the fact that the semigroup defined by González Pérez does not depend on the coordinate system (Corollary 3.4). First, in fixed quasi-ordinary coordinates, we restrict to series which have dominating terms and we define a semigroup (Definition 2.3) which can be seen to coincide with the one of González Pérez. Then we construct another semigroup (Definition 3.2) which imitates the intrinsic one given before in the case of curves, and we show the isomorphism of the two semigroups (Theorem 3.3). Moreover, we associate values in the semigroup to some elements in the local ring of the quasi-ordinary germ of surface (Corollary 3.5). From the invariance of the semigroup we deduce the analytic invariance of the normalized characteristic exponents (Corollary 3.7). Other proofs of this fact were given by Lipman [8,9] and Luengo Velasco [11,12].

Detailed proofs of our results are given in [14].

2. Basic definitions

Let $d \geqslant 1$ be an integer. Define the *algebra of fractional series* $\widetilde{\mathbf{C}\{X\}} := \lim_{\substack{\rightarrow \\ N \geqslant 0}} \mathbf{C}\{X_1^{1/N}, \ldots, X_d^{1/N}\}$, where $X := (X_1, \ldots, X_d)$. If $\eta \in \widetilde{\mathbf{C}\{X\}}$ can be written $\eta = X^m u(X)$, with $m \in \mathbf{Q}_+^d$ and $u \in \widetilde{\mathbf{C}\{X\}}$, $u(0, \ldots, 0) \neq 0$, we say that η has a *dominating term* and we define its *dominating exponent* to be $v_X(\eta) := m$.

Let $\mathcal{A}$ be a reduced equi-dimensional local complex-analytic algebra of dimension d and $(\mathcal{S}, 0)$ a germ of complex space such that $\mathcal{O}_{\mathcal{S},0} \simeq \mathcal{A}$.

DEFINITION 2.1. – The algebra $\mathcal{A}$ and the germ $(\mathcal{S}, 0)$ are called *quasi-ordinary* if there exists a finite morphism ψ from $(\mathcal{S}, 0)$ to a smooth space of the same dimension, whose discriminant is contained in a hypersurface with normal crossings. Such a morphism ψ is called *quasi-ordinary*.

All germs of curves are quasi-ordinary. A germ of surface whose local algebra is isomorphic to $\mathbf{C}\{X_1, X_2, Y\}/(Y^2 - X_1X_2)$ is quasi-ordinary, we call it *ordinary quadratic*.

Quasi-ordinary germs appear naturally in the Jung method of resolution of the singularities of a germ by embedded resolution of the discriminant locus of a finite morphism from the germ to a smooth space of same dimension (*see* [7] for the case of dimension 2). Quasi-ordinary hypersurface germs were first systematically studied by Lipman [8] when $d = 2$, see also the survey [9]. This study was extended to any $d \geqslant 2$ in [10].

In the special case in which $\mathcal{S}$ is of embedding dimension $d + 1$, one can find an element Y in the maximal ideal of $\mathcal{A}$ and local coordinates X on the target space of ψ such that (ψ, Y) embeds $(\mathcal{S}, 0)$ in $\mathbf{C}^d \times \mathbf{C}$. By the Weierstrass preparation theorem, the image of $\mathcal{S}$ by (ψ, Y), identified in the sequel with $\mathcal{S}$, is defined by a unitary polynomial $f \in \mathbf{C}\{X\}[Y]$. The discriminant locus of ψ is defined by the discriminant $\Delta_Y(f)$ of f. So one can choose the coordinates X in such a way that $\Delta_Y(f)$ has a dominating term.

DEFINITION 2.2. – Let $f \in \mathbf{C}\{X\}[Y]$ be unitary. If $\Delta_Y(f)$ has a dominating term, we say that f is *quasi-ordinary*. If $\mathcal{A} \simeq \mathbf{C}\{X\}[Y]/(f)$, with f quasi-ordinary, we say that f is a *qo-defining polynomial* of $\mathcal{S}$ and of the algebra $\mathcal{A}$.

By the theorem of Jung–Abhyankar (*see* [1,10]), which generalizes the theorem of Newton–Puiseux, if $f \in \mathbf{C}\{X\}[Y]$ is quasi-ordinary, then the set $R(f)$ of roots of f embeds canonically in the algebra $\widetilde{\mathbf{C}\{X\}}$. In the sequel, we consider always $R(f)$ as a subset of $\widetilde{\mathbf{C}\{X\}}$.

A difficulty for extending the plane branch definition of the semigroup is that in dimension > 1, fractional series may have no dominating term. *One way to force the existence of a dominating term is to restrict to those functions which do have one*.

Suppose that f is irreducible. Let $\xi \in R(f)$. We define the following multiplicative subsemigroup of $\mathbf{C}\{X\}[Y] - (f)$:

$$\mathcal{D}(f) := \big\{h \in \mathbf{C}\{X\}[Y] - (f),\ h(\xi) \text{ has a dominating term}\big\}.$$

This semigroup is independent of the chosen ξ.

DEFINITION 2.3. – The *semigroup of $\mathcal{A}$ with respect to f*, denoted $\Gamma(f)$, is the following subsemigroup of $(\mathbf{Q}_+^d, +)$: $\Gamma(f) = \{v_X(h(\xi)),\ h \in \mathcal{D}(f)\}$.

All the differences of roots of f have dominating terms, whose dominating exponents are totally ordered for the componentwise order [8,10]. We call them the *characteristic exponents* of f and we denote them by

$A_1 < \cdots < A_G$, $A_i = (A_i^1, \ldots, A_i^d)$, $\forall i \in \{1, \ldots, G\}$. After possibly permuting the variables $X_1, \ldots, X_d$, we can suppose that $(A_1^1, \ldots, A_G^1) \geqslant_{\text{lex}} \cdots \geqslant_{\text{lex}} (A_1^d, \ldots, A_G^d)$. We say that f is *normalized* with respect to $\mathcal{S}$ if either $A_1^2 \neq 0$ or $A_1^1 > 1$. Lipman [8] proved that any irreducible quasi-ordinary germ of hypersurface has normalized qo-defining polynomials, see also [9,10] and [5].

Following [10], we define inductively the abelian groups $M_0 := \mathbf{Z}^d$, $M_k := M_{k-1} + \mathbf{Z}A_k$, $\forall k \in \{1, \ldots, G\}$ and the successive indices $N_k := \text{card}(M_k/M_{k-1})$, $\forall k \in \{1, \ldots, G\}$. Following [5,6] we define the vectors $\overline{A}_1 := A_1$, $\overline{A}_k := N_{k-1}\overline{A}_{k-1} + A_k - A_{k-1}$, $\forall k \in \{2, \ldots, G\}$, $\overline{A}_{G+1} := \infty$.

Another way to force the existence of a dominating term is to generalize that notion. In [5], González Pérez considered instead the Newton polyhedra $\mathcal{N}_X(\eta)$ of $\eta \in \widehat{\mathbf{C}\{X\}}$ (*see* Section 4). He proved that the set of vertices of $\mathcal{N}_X(h(\xi))$, where h varies through $\mathbf{C}\{X\}[Y] - (f)$, is a semigroup equal to $\mathbf{N}^d + \mathbf{N}\overline{A}_1 + \cdots + \mathbf{N}\overline{A}_G$. The definition of $\Gamma(f)$ we give leads to the same semigroup (*see* [14]).

3. The results

In the sequel we suppose that $(\mathcal{S}, 0)$ is irreducible, quasi-ordinary of dimension 2 and embedding dimension 3. Moreover, we suppose that 0 is not smooth on $\mathcal{S}$. Let f be a qo-defining polynomial of $\mathcal{S}$. With our hypothesis, f is irreducible.

Define $\nu : \overline{\mathcal{S}} \to \mathcal{S}$ to be the normalisation morphism of the surface $\mathcal{S}$. The germ $\overline{\mathcal{S}}$ is then of Jung–Hirzebruch type (*see* [3]). Let $\mu : \mathcal{R} \to \overline{\mathcal{S}}$ be the minimal resolution of the singular locus of $\overline{\mathcal{S}}$, which is either empty or a point. Define also $\eta : \overline{\mathcal{R}} \to \mathcal{R}$ to be either the blow-up morphism of the point $\nu^{-1}(0)$ if $(\nu \circ \mu)^{-1}(\text{Sing}(\mathcal{S}))$ is smooth, or the identity on $\mathcal{R}$ otherwise. Here $\text{Sing}(\mathcal{S})$ denotes the singular locus of $\mathcal{S}$. In general it is not isolated. In [8,10], Lipman describes it in terms of the characteristic exponents of any qo-defining polynomial of $\mathcal{S}$.

Define $\theta := \nu \circ \mu \circ \eta : \overline{\mathcal{R}} \to \mathcal{S}$. Let θ^* be the corresponding morphism of local algebras. Clearly, θ depends only on the analytic type of $\mathcal{S}$ and not on any particular qo-defining polynomial.

LEMMA 3.1. – *If the germ $\mathcal{S}$ is not ordinary quadratic, the subset $\theta^{-1}(\text{Sing}(\mathcal{S}))$ of the smooth surface $\overline{\mathcal{R}}$ is a divisor with normal crossings which has at least one singular point.*

Suppose in the sequel that $\mathcal{S}$ *is not ordinary quadratic* and that P is a singular point of $\theta^{-1}(\text{Sing}(\mathcal{S}))$. Take local coordinates $X^P := (X_1^P, X_2^P)$ at P that define the germ $\theta^{-1}(\text{Sing}(\mathcal{S}))_P$. Then the following multiplicative subsemigroup of the algebra $\mathcal{A}$ does not depend on the choice of X^P:

$$\mathcal{D}_P(\mathcal{S}) := \{h \in \mathcal{A} - \{0\},\ \theta^*(h)_P \text{ has a dominating term in the coordinates } X^P\}.$$

The following definition generalizes the first one presented in Section 1 for plane curves:

DEFINITION 3.2. – The *semigroup of $\mathcal{S}$ with respect to P*, denoted $\Gamma_P(\mathcal{S})$, is the following subsemigroup of $(\mathbf{N}^2, +)$: $\Gamma_P(\mathcal{S}) := \{v_{X^P}(\theta^*(h)_P),\ h \in \mathcal{D}_P(\mathcal{S})\}$.

Our main result is:

THEOREM 3.3. – *Suppose that the germ $\mathcal{S}$ is not ordinary quadratic. Let f be a quasi-ordinary defining polynomial of $\mathcal{S}$. For every singular point P of the reduced divisor $\theta^{-1}(\text{Sing}(\mathcal{S}))$, the image of the restriction mapping $\mathcal{D}(f) \to \mathcal{A}$ is contained in the set $\mathcal{D}_P(\mathcal{S})$. It induces a well-defined mapping which realises an isomorphism of semigroups*:

$$\begin{aligned} \Phi_P : \quad \Gamma(f) \quad &\longrightarrow \Gamma_P(\mathcal{S}), \\ v_X(h(\xi)) &\longrightarrow v_{X^P}\big(\theta^*(h|_{\mathcal{S}})_P\big). \end{aligned}$$

The case of an ordinary quadratic germ can be easily treated separately (*see* [14]).

As the left-hand semigroup does not depend on the choice of P and the right-hand one does not depend on the choice of qo-defining f, we get:

COROLLARY 3.4. – *As an abstract semigroup, $\Gamma(f)$ does not depend on the chosen defining polynomial f of $\mathcal{S}$. We call it the semigroup of $\mathcal{S}$, denoted $\Gamma(\mathcal{S})$.*

As a by-product of the proof of Theorem 3.3, we get a way to associate to some elements of $\mathcal{A}$ a value in the semigroup $\Gamma(\mathcal{S})$:

COROLLARY 3.5. – *Let f be a qo-defining polynomial of the germ $\mathcal{S}$ and $\xi \in R(f)$. If $h \in \mathcal{D}(f)$, then the dominating exponent $v_X(h(\xi))$, seen as an element of the abstract semigroup $\Gamma(\mathcal{S})$, depends only on the image $h\,|_{\mathcal{S}} \in \mathcal{A}$, and not on the choice of f.*

If $\mathrm{Sing}(\mathcal{S})$ has two irreducible components, then the image of $\mathcal{D}(f)$ in $\mathcal{A}$ obtained by restriction is independent of the defining f. This fact is no longer true if $\mathrm{Sing}(\mathcal{S})$ is irreducible (look at $f := Y^2 - X_1 X_2^3$, $h := Y \in \mathcal{D}(f)$ and change the quasi-ordinary projection).

Let $G(\mathcal{S})$ be the group generated by $\Gamma(\mathcal{S})$ and $E(\mathcal{S}) := G(\mathcal{S}) \otimes_{\mathbf{Z}} \mathbf{R}$ be the real vector space generated by it. Denote by $\sigma(\mathcal{S})$ the convex cone generated by $\Gamma(\mathcal{S})$ in $E(\mathcal{S})$, the union of the nonnegative real combinations of elements of $\Gamma(\mathcal{S})$. It is strictly convex. Let u^1 and u^2 be the smallest non-zero elements of $\Gamma(\mathcal{S})$ situated on the two edges of $\sigma(\mathcal{S})$. The following lemma shows that one can extract canonically vectors of $\mathbf{Q}_+^2$ from the isomorphism type of $\Gamma(\mathcal{S})$:

LEMMA 3.6. – *For any $j \geqslant 1$, if $\alpha_1, \ldots, \alpha_{j-1}$ are already defined and verify $\Gamma(\mathcal{S}) \neq \mathbf{N}u^1 + \mathbf{N}u^2 + \mathbf{N}\alpha_1 + \cdots + \mathbf{N}\alpha_{j-1}$, there exists a unique smallest element α_j of $\Gamma(\mathcal{S})$ not contained in the semi-group $\mathbf{N}u^1 + \mathbf{N}u^2 + \mathbf{N}\alpha_1 + \cdots + \mathbf{N}\alpha_{j-1}$. After possibly permuting u^1 and u^2, the components of $\alpha_1, \ldots, \alpha_g$ written in the basis u^1, u^2 coincide with the vectors $\overline{A}_1, \ldots, \overline{A}_G$ of $\mathbf{Q}_+^2$ associated to any normalized qo-defining polynomial of $\mathcal{S}$.*

Using this lemma, by generalizing the method used in [13] for plane curves, we get another corollary of Theorem 3.3:

COROLLARY 3.7. – *The characteristic exponents of a normalized qo-defining polynomial f of the germ $\mathcal{S}$ do not depend on the choice of f.*

4. Expansions according to semiroots

One of the main tools in the proof of Theorem 3.3 is Lemma 4.3 below, which generalizes the properties of Abhyankar's expansions in terms of semiroots in the plane branch case (*see* [2] and [13]). Semiroots were introduced in the quasi-ordinary case by González Pérez in [5].

DEFINITION 4.1. – Let us fix $\xi \in R(f)$. Take any $k \in \{0, \ldots, G\}$. A unitary polynomial $f_k \in \mathbf{C}\{X\}[Y]$ is called a *k-semiroot* of f if f_k is of degree $N_1 \cdots N_k$, and $f_k \in \mathcal{D}(f)$ with $v_k(f_k(\xi)) = \overline{A}_{k+1}$. A $(G+1)$-tuple $(f_0, \ldots, f_G)$ such that $\forall k \in \{0, \ldots, G\}$, f_k is a k-semiroot of f is called a *complete system of semiroots* for f. These objects are independent of the choice of ξ.

Let $(f_0, \ldots, f_G)$ be a complete system of semiroots for f (which always exists, for example the minimal polynomials of suitable truncations of ξ, *see* [5]). Generalizing [2], any $h \in \mathbf{C}\{X\}[Y]$ can be uniquely written as $h = \sum c_{i_0 \cdots i_G} f_0^{i_0} \cdots f_G^{i_G}$, the summation being done over the $(G+1)$-tuples $(i_0, \ldots, i_G) \in \mathbf{N}^{G+1}$, with $0 \leqslant i_k \leqslant N_{k+1} - 1$, $\forall k \in \{0, \ldots, G\}$ (where $N_{G+1} := +\infty$) and $c_{i_0 \cdots i_G} \in \mathbf{C}\{X\}$.

DEFINITION 4.2. – The preceding equality is called *the* $(f_0, \ldots, f_G)$*-adic expansion* of h. The set $\{(i_0, \ldots, i_G), c_{i_0 \cdots i_G} \neq 0\}$ is called *the* $(f_0, \ldots, f_G)$*-adic support of* h, denoted $\mathrm{Supp}_{(f_0, \ldots, f_G)}(h)$.

Such expansions are the main tool in the proofs of the results quoted in the last paragraph of Section 2.

If $\eta \in \widetilde{\mathbf{C}\{X\}}$, we define its *Newton polyhedron* $\mathcal{N}_X(\eta)$ to be the convex hull in $\mathbf{R}^d$ of the set $\mathrm{Supp}_X(\eta) + \mathbf{R}_+^d$, where $\mathrm{Supp}_X(\eta)$ denotes the support of η written as a series is the variables X.

LEMMA 4.3. – *If $h = \sum c_{i_0 \cdots i_G} f_0^{i_0} \cdots f_G^{i_G}$ is the $(f_0, \ldots, f_G)$-adic expansion of $h \in \mathbf{C}\{X\}[Y]$, then for every $\xi \in R(f)$, the set of vertices of the Newton polyhedra $\mathcal{N}_X(c_{i_0 \cdots i_G}(f_0(\xi))^{i_0} \cdots (f_G(\xi))^{i_G})$ are pairwise disjoint, when $(i_0, \ldots, i_G)$ varies through the $(f_0, \ldots, f_G)$-adic support of h.*

5. Sketch of the proof of Theorem 3.3

Denote $\overline{\psi} := \psi \circ \theta : \overline{\mathcal{R}} \to \mathbf{C}^2$.

As the image of $Y \in \mathbf{C}\{X\}[Y]$ in $\mathcal{A}$ verifies the equation $f(X_1, X_2, Y) = 0$, one sees that $Y\mid_{\mathcal{S}}$ can be thought as an element of $R(f)$. Denoting it by ξ, one has the equality:

$$\overline{\psi}^*\bigl(h(\xi)\bigr) = \theta^*(h\mid_{\mathcal{S}}), \quad \forall h \in \mathbf{C}\{X\}[Y]. \tag{1}$$

One can choose as representative of $\overline{\psi}$ a localisation to an open set of a toric morphism (*see* [5] and [14]). Indeed, one can construct a normalisation morphism ν such that $\psi \circ \nu$ is the localisation to an open set of a toric morphism. Then, μ and η can also be realised as toric morphisms. With such representatives of the morphisms, the point P is an orbit of dimension 0 and one can choose for X^P canonical toric coordinates. With such coordinates, the morphism $\overline{\psi}^*$ is monomial, and using formula (1), we see that Φ_P is *injective*.

In order to prove its *surjectivity*, we must show that if $h \in \mathbf{C}\{X\}[Y]$ verifies $h\mid_{\mathcal{S}} \in \mathcal{D}_P(\mathcal{A})$, then one can find another element $h' \in \mathcal{D}(f)$ such that $v_{X^P}(\theta^*(h'\mid_{\mathcal{S}})_P) = v_{X^P}(\theta^*(h\mid_{\mathcal{S}})_P)$. As $f\mid_{\mathcal{S}} = 0$, we can suppose that $\deg(h) < \deg(f)$, after possibly making the euclidian division of h by f. We consider then a complete system $(f_0, \ldots, f_G)$ of semiroots of f and the $(f_0, \ldots, f_G)$-adic expansion of h, which by our hypothesis is of the form $h = \sum c_{i_0 \cdots i_{G-1}} f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}$. Using Lemma 4.3, we see that there exists a tuple $(i_0, \ldots, i_{G-1}, 0) \in \mathrm{Supp}_{(f_0,\ldots,f_G)}(h)$ such that $v_{X^P}(\theta^*(h\mid_{\mathcal{S}})_P) = v_{X^P}(\theta^*(c_{i_0\cdots i_{G-1}} f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}\mid_{\mathcal{S}})_P) = v_{X^P}(\theta^*(X^m f_0^{i_0} \cdots f_{G-1}^{i_{G-1}}\mid_{\mathcal{S}})_P)$, where m is one of the vertices of the Newton polygon $\mathcal{N}_X(c_{i_0\cdots i_{G-1}})$. But $X^m f_0^{i_0} \cdots f_{G-1}^{i_{G-1}} \in \mathcal{D}(f)$, which proves that Φ_P is surjective.

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References

[1] S.S. Abhyankar, On the ramification of algebraic functions, Amer. J. Math. 77 (1955) 575–592.
[2] S.S. Abhyankar, Expansion Techniques in Algebraic Geometry, Tata Institute of Fundamental Research, Bombay, 1977.
[3] W. Barth, C. Peters, A.D.M. Van de Ven, Compact Complex Surfaces, Springer-Verlag, 1984.
[4] Y.-N. Gau, Embedded topological classification of quasi-ordinary singularities, Mem. Amer. Math. Soc. 388 (1988).
[5] P. González Pérez, Quasi-ordinary singularities via toric geometry, Thesis, University of La Laguna, Tenerife, Spain, September 2000.
[6] P. González Pérez, The semigroup of a quasi-ordinary hypersurface, submitted.
[7] H.B. Laufer, Normal Two-Dimensional Singularities, Princeton University Press, 1971.
[8] J. Lipman, Quasi-ordinary singularities of embedded surfaces, Thesis, Harvard University, 1965.
[9] J. Lipman, Quasi-ordinary singularities of surfaces in $\mathbf{C}^3$, in: Proc. Sympos. Pure Math., Vol. 40, 1983, Part 2, pp. 161–172.
[10] J. Lipman, Topological invariants of quasi-ordinary singularities, Mem. Amer. Math. Soc. 388 (1988).
[11] I. Luengo Velasco, Sobre la estructura de las singularidades de las superficies algebroides sumergidas, Thesis, Univ. Complutense, Madrid, September 1979.
[12] I. Luengo Velasco, On the structure of embedded algebroid surfaces, in: Proc. Sympos. Pure Math., Vol. 40, 1983, Part 2, pp. 185–192.
[13] P. Popescu-Pampu, Approximate Roots, Proceedings of the International Conference and Workshop on Valuation Theory Saskatoon, Canada, 1999, to appear.
[14] P. Popescu-Pampu, Arbres de contact des singularités quasi-ordinaires et graphes d'adjacence pour les 3-variétés réelles, Thesis, University Paris 7, November 2001.
[15] O. Zariski, Le problème des modules pour les branches planes, Hermann, 1986.