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Équations aux dérivées partielles/*Partial Differential Equations*

Explicit limits for nonperiodic homogenization and reduction of dimension

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Abstract The aim of this Note is to give explicit limit expressions, for diffusion equations involving a small parameter ε, describing both nonperiodic homogenization and reduction of dimension. We consider two kinds of reduction of dimension: the case of plates and the case of thin cylinders. In particular, we give the limit diffusion equation for stratified plates. This is completely explicit and requires no special assumption, except stratification. In the case of thin cylinders, the formulae are less explicit, but we also indicate some simple applications to fibered materials. ***To cite this article: B. Gustafsson, J. Mossino, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 977–982.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Limites explicites pour l'homogénéisation non-périodique et la réduction de dimension

Résumé Le but de cette Note est d'indiquer les limites explicites d'équations de diffusion comprenant un petit paramètre ε, décrivant à la fois l'homogénéisation non-périodique et la réduction de dimension. Nous considérons deux cas de réduction de dimension : les plaques et les cylindres minces. En particulier nous donnons l'équation limite explicite pour les plaques stratifiées. Dans le cas des cylindres minces, les formules sont moins explicites, mais nous mentionnons également des applications aux matériaux fibrés. ***Pour citer cet article : B. Gustafsson, J. Mossino, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 977–982.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans cette Note, Ω^ε est un domaine mince de $\mathbb{R}^N$ $(N \geqslant 2)$, représentant soit une plaque horizontale, soit un cylindre mince vertical, et l'on considère l'équation de la diffusion dans Ω^ε, avec des conditions au bord mixtes, de type Dirichlet–Neumann.

Le point générique de $\mathbb{R}^N$ est noté $x = (x', x_N) = (x_1, \dots, x_{N-1}, x_N)$. Les coefficients de l'équation composent une matrice de conductivité notée $A^\varepsilon = A^\varepsilon(x', x_N/\varepsilon)$, pour $x = (x', x_N)$ dans la plaque, ou bien $A^\varepsilon = A^\varepsilon(x'/\varepsilon, x_N)$, pour x dans le cylindre mince. Dans les deux cas $A^\varepsilon = A^\varepsilon(x)$, pour x dans un cylindre fixe Ω.

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Par changement de variable, l'équation de la diffusion, écrite sous forme variationnelle dans Ω, est

$$
\begin{aligned}
&u^\varepsilon \in \mathrm{H}^1(\Omega),\ u^\varepsilon = 0 \quad \text{sur } \Gamma_D \quad \text{et} \quad \forall v \in \mathrm{H}^1(\Omega), \quad v = 0 \quad \text{sur } \Gamma_D,\\
&\int_\Omega \left[A^\varepsilon(x)\nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon v\right] \mathrm{d}x = \int_\Omega f^\varepsilon v \,\mathrm{d}x + \int_\Omega \left[g^\varepsilon, \nabla^\varepsilon v\right] \mathrm{d}x + \int_{\Gamma_N} h^\varepsilon v \,\mathrm{d}\gamma,
\end{aligned}
\tag{0.1}
$$

où $\partial\Omega = \Gamma_D \cup \Gamma_N$, les conditions de Dirichlet et Neumann étant satisfaites respectivement sur Γ_D et Γ_N ; $[\cdot,\cdot]$ dénote le produit scalaire dans $\mathbb{R}^N$ et l'opérateur ∇^ε prend deux formes différentes :

$$
\begin{aligned}
\nabla^\varepsilon v &= \left(\nabla' v, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_N}\right) = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_{N-1}}, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_N}\right) \quad \text{pour la plaque,}\\
\nabla^\varepsilon v &= \left(\frac{1}{\varepsilon}\nabla' v, \frac{\partial v}{\partial x_N}\right) = \left(\frac{1}{\varepsilon}\frac{\partial v}{\partial x_1}, \dots, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_{N-1}}, \frac{\partial v}{\partial x_N}\right) \quad \text{pour le cylindre mince.}
\end{aligned}
$$

Le passage à la limite dans (0.1), lorsque ε tend vers zéro, combine les difficultés bien connues de l'homogénéisation et de la réduction de dimension (*cf.* e.g. [1,17,22,2–6,13–16,19]). Dans cette Note, en supposant que A^ε satisfait certaines hypothèses de compacité par compensation ne requérant aucune périodicité, nous donnons la forme explicite de la H-limite de la suite A^ε, notée A, et nous montrons que le problème limite de (0.1) prend la forme

$$
\int_\Omega \left[A\nabla''(u,y), \nabla''(v,z)\right] \mathrm{d}x = \int_\Omega f v \,\mathrm{d}x + \int_\Omega \left[g, \nabla''(v,z)\right] \mathrm{d}x + \int_{\Gamma_N} h v \,\mathrm{d}\gamma,
\tag{0.2}
$$

où l'opérateur ∇'' est défini par

$$
\begin{aligned}
\nabla''(v,z) &= \left(\nabla' v, \frac{\partial z}{\partial x_N}\right) \quad \text{pour la plaque,}\\
\nabla''(v,z) &= \left(\nabla' z, \frac{\partial v}{\partial x_N}\right) \quad \text{pour le cylindre mince.}
\end{aligned}
$$

Ce résultat n'est pas vrai en général (*cf.* [20,21,12]).

La preuve utilise la méthode classique de compacité par compensation [22], appliquée à une décomposition convenable de A^ε : $A^\varepsilon = (M^\varepsilon)^{-1} P^\varepsilon$, avec deux expressions différentes du couple $(M^\varepsilon, P^\varepsilon)$, l'une pour la plaque, l'autre pour le cylindre mince. L'application la plus simple concerne les plaques stratifiées, mais nous mentionnons des applications aux cylindres minces constitués de matériaux fibrés (*cf.* aussi [12]).

1. Nonperiodic homogenization and reduction of dimension for plates

Let ω be a bounded domain in $\mathbb{R}^{N-1}$, $N \geqslant 2$, and let Ω be the cylinder $\Omega = \omega \times (-\frac{1}{2}, \frac{1}{2})$. Now, for ε ($\leqslant 1$) taking a sequence of values tending to zero, Ω^ε ($\subset \Omega$) represents the horizontal plate, that is the flat cylinder $\Omega^\varepsilon = \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, with lateral boundary $\Sigma^\varepsilon = \partial\omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$.

Let $A^\varepsilon : \Omega \to \mathbb{R}^{N\times N}$ be a sequence of (not necessarily symmetric) matrices with L^∞-coefficients, such that

$$
\exists \alpha, \beta, 0 < \alpha \leqslant \beta,\ \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N, \quad \left[A^\varepsilon(x)\xi, \xi\right] \geqslant \alpha|\xi|^2 \quad \text{and} \quad \left|A^\varepsilon(x)\xi\right| \leqslant \beta|\xi|,
\tag{1.3}
$$

where $[\cdot,\cdot]$ denotes the scalar product in $\mathbb{R}^N$ and $|\cdot|$ denotes the Euclidian norm.

We consider the variational problem, with given data $F^\varepsilon \in \mathrm{L}^2(\Omega^\varepsilon)$, $G^\varepsilon \in \mathrm{L}^2(\Omega^\varepsilon)^N$, h_+^ε and $h_-^\varepsilon \in \mathrm{H}^{-1/2}(\omega)$,

$$
\begin{aligned}
&U^\varepsilon \in \mathcal{V}^\varepsilon = \left\{V \in \mathrm{H}^1(\Omega^\varepsilon),\ V = 0 \text{ on } \Sigma^\varepsilon\right\} \quad \text{and} \quad \forall V \in \mathcal{V}^\varepsilon,\\
&\int_{\Omega^\varepsilon} \left[A^\varepsilon\left(x', \frac{x_N}{\varepsilon}\right)\nabla U^\varepsilon, \nabla V\right] \mathrm{d}x = \int_{\Omega^\varepsilon} F^\varepsilon V \,\mathrm{d}x + \int_{\Omega^\varepsilon} \left[G^\varepsilon, \nabla V\right] \mathrm{d}x
\end{aligned}
$$

$$+\int_\omega \varepsilon h_+^\varepsilon(x')V\left(x',\frac{\varepsilon}{2}\right)\mathrm{d}x' + \int_\omega \varepsilon h_-^\varepsilon(x')V\left(x',-\frac{\varepsilon}{2}\right)\mathrm{d}x', \tag{1.4}$$

where the generic point in $\mathbb{R}^N$ is denoted by $x=(x',x_N)=(x_1,\dots,x_{N-1},x_N)$. For convenience, since this is not restrictive, we have choosed to write the coefficients in the form $A^\varepsilon(x',x_N/\varepsilon)$ instead of $A^\varepsilon(x)$. It is classical that (1.4) admits a unique solution U^ε, solving (in a weak sense) a standard diffusion equation.

Such problems were considered by A. Damlamian and M. Vogelius in [10], with symmetric matrices A^ε. They proved that, up to extraction of a subsequence, there exists a symmetric matrix $A^0:\omega\to\mathbb{R}^{(N-1)\times(N-1)}$, such that $U^\varepsilon(x',\varepsilon x_N)$ converges weakly in $\mathrm{H}^1(\Omega)$ to the solution $U=U(x')$ of a $(N-1)$-dimensional problem defined in terms of A^0, provided the data converge in a natural sense. As it is the case for H-convergence, the limit matrix A^0 does not depend on the source and boundary data. But of course, A^0 is not the H-limit of A^ε, since A^0 has size $(N-1)\times(N-1)$ and is defined in ω, while A^ε is a $N\times N$-matrix, defined in Ω. Moreover, no explicit expression of A^0 was known, except if A^ε is a periodic function depending on x' only (*see* [2,3,13]).

By using Theorem 1 below, it is easy to prove (*cf.* Remarks 1 and 2) that A^0 is explicit in terms of the H-limit A of A^ε, under some compensated compactness condition, which requires no periodicity and generalizes the stratified case $A^\varepsilon=A^\varepsilon(x_N)$.

More precisely, we write

$$A^\varepsilon=\begin{pmatrix}(A^\varepsilon)' & C^\varepsilon\\ L^\varepsilon & a^\varepsilon_{NN}\end{pmatrix}=\left(M^\varepsilon\right)^{-1}P^\varepsilon, \tag{1.5}$$

with $(A^\varepsilon)'_{ij}=a^\varepsilon_{ij}$, $C^\varepsilon_i=a^\varepsilon_{iN}$, $L^\varepsilon_j=a^\varepsilon_{Nj}$, for $i,j<N$, and with

$$M^\varepsilon=\begin{pmatrix}I & -C^\varepsilon/a^\varepsilon_{NN}\\ 0 & 1/a^\varepsilon_{NN}\end{pmatrix},\qquad P^\varepsilon=\begin{pmatrix}(A^\varepsilon)'-C^\varepsilon L^\varepsilon/a^\varepsilon_{NN} & 0\\ L^\varepsilon/a^\varepsilon_{NN} & 1\end{pmatrix}. \tag{1.6}$$

We denote by P_i and M_i the ith line (i.e., row) vectors of P and M, respectively, and define $\operatorname{div}P$ and $\operatorname{curl}M$ by $(\operatorname{div}P)_i=\operatorname{div}P_i$ and $(\operatorname{curl}M)_{ijk}=(\operatorname{curl}M_i)_{jk}$. Though the decomposition of A^ε will be different in the next section, in the whole note the compensated compactness condition is the one introduced by P. Courilleau in [7] (*see* also [11,8,9]), namely

$$\begin{aligned}&\left\{\operatorname{curl}M^\varepsilon\right\}_\varepsilon \text{ is relatively compact in } \mathrm{H}^{-1}(\Omega)^{N\times N\times N}\quad\text{and}\\ &\left\{\operatorname{div}P^\varepsilon\right\}_\varepsilon \text{ is relatively compact in } \mathrm{H}^{-1}(\Omega)^{N},\end{aligned} \tag{1.7}$$

and the convergence condition is that

$$M^\varepsilon\rightharpoonup M\quad\text{and}\quad P^\varepsilon\rightharpoonup P,\quad\text{weakly* in } \mathrm{L}^\infty(\Omega)^{N\times N}. \tag{1.8}$$

THEOREM 1. – *Assume* (1.3) *and* (1.7), *with* M^ε *and* P^ε *defined in* (1.5) *and* (1.6). *Then the* H-*limit of* A^ε *is characterized by* $A=M^{-1}P$, *with* M *and* P *given by* (1.8), *which we assume in addition. Define* f^ε *and* g^ε *from* F^ε *and* G^ε *by* $f^\varepsilon(x)=F^\varepsilon(x',\varepsilon x_N)$, $g^\varepsilon(x)=G^\varepsilon(x',\varepsilon x_N)$. *Assume moreover that the data converge in the following sense*:

$$\begin{cases} f^\varepsilon\rightharpoonup f\quad \textit{weakly in } \mathrm{L}^2(\Omega), & g^\varepsilon\rightharpoonup g\quad \textit{weakly in } \mathrm{L}^2(\Omega)^N,\\ h_+^\varepsilon\rightharpoonup h_+\quad\textit{and}\quad h_-^\varepsilon\rightharpoonup h_-\quad \textit{weakly in } \mathrm{H}^{-1/2}(\omega) \end{cases} \tag{1.9}$$

and assume also that

$$\left\{\frac{\partial g_N^\varepsilon}{\partial x_N}\right\}_\varepsilon \textit{ is relatively compact in } \mathrm{H}^{-1}(\Omega). \tag{1.10}$$

Let $u^\varepsilon(x) = U^\varepsilon(x', \varepsilon x_N)$, *where* U^ε *is the solution of* (1.4) *and let* ∇^ε *be the operator defined by* $\nabla^\varepsilon v = \big(\nabla' v, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_N}\big) = \big(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_{N-1}}, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_N}\big)$. *Then, when* ε *tends to zero,*

$$\begin{cases} u^\varepsilon \rightharpoonup u \quad \text{weakly in } \mathrm{H}^1(\Omega), \quad \frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial x_N} \rightharpoonup \frac{\partial y}{\partial x_N} \quad \text{weakly in } \mathrm{L}^2(\Omega), \\ \sigma^\varepsilon = A^\varepsilon \nabla^\varepsilon u^\varepsilon \rightharpoonup \sigma = A \begin{pmatrix} \nabla' u \\ \partial y/\partial x_N \end{pmatrix} \quad \text{weakly in } \mathrm{L}^2(\Omega)^N, \end{cases} \tag{1.11}$$

where u *depends only on* x' *and* (u, y) *is the unique solution of the limit variational problem*:

$$u \in \mathrm{H}^1_0(\omega),\ y \in \mathrm{L}^2\left(\omega; \mathrm{H}^1_m\left(-\frac{1}{2}, \frac{1}{2}\right)\right) \quad \text{and} \quad \forall v \in \mathrm{H}^1_0(\omega),\ \forall z \in \mathrm{L}^2\left(\omega; \mathrm{H}^1_m\left(-\frac{1}{2}, \frac{1}{2}\right)\right),$$

$$\int_\Omega \left[A\begin{pmatrix} \nabla' u \\ \partial y/\partial x_N \end{pmatrix}, \begin{pmatrix} \nabla' v \\ \partial z/\partial x_N \end{pmatrix}\right] \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x + \int_\Omega \left[g, \begin{pmatrix} \nabla' v \\ \partial z/\partial x_N \end{pmatrix}\right] \mathrm{d}x + \int_\omega (h_+ + h_-) v \, \mathrm{d}x', \tag{1.12}$$

H^1_m *denoting the subset of functions of* H^1, *having mean value zero.*

Remark 1. – By eliminating y in profit of u, it is easy to see that u is the unique solution of the variational problem

$$\int_\omega \left[A^0 \nabla' u, \nabla' v\right] \mathrm{d}x' = \int_\omega f^0 v \, \mathrm{d}x' + \int_\omega \left[g^0, \nabla' v\right] \mathrm{d}x' + \int_\omega (h_+ + h_-) v \, \mathrm{d}x',$$

where $a^0_{ij} = m(p_{ij})$, $f^0 = m(f)$, $g^0_i = m(g_i - \frac{a_{iN}}{a_{NN}} g_N)$, m denoting the mean value over the x_N-direction. Then y is given by $m(y) = 0$ and

$$\frac{\partial y}{\partial x_N} = \frac{1}{a_{NN}}\left(g_N - \sum_{j<N} a_{Nj}\frac{\partial u}{\partial x_j}\right) = \frac{1}{a_{NN}}\big(g_N - [L, \nabla' u]\big).$$

Remark 2. – In the case of stratified plates, $A^\varepsilon = A^\varepsilon(x_N)$, condition (1.7) is automatically satisfied, since curl $M^\varepsilon \equiv 0$ and div $P^\varepsilon \equiv 0$. Moreover (1.8) are the classical formulae giving the H-limit of A^ε (*see*, e.g., [17]).

2. Nonperiodic homogenization and reduction of dimension for thin cylinders

As in Section 1, ω is a bounded domain in $\mathbb{R}^{N-1}$, $N \geqslant 2$. Now Ω denotes the cylinder $\Omega = \omega \times (0, 1)$ and for ε small, Ω^ε represents the vertical thin cylinder $\Omega^\varepsilon = \varepsilon\omega \times (0, 1)$, with lateral boundary $\Sigma^\varepsilon = \varepsilon\partial\omega \times (0, 1)$ and horizontal bases $\Gamma^\varepsilon = \Gamma^\varepsilon_- \cup \Gamma^\varepsilon_+ = (\varepsilon\omega \times \{0\}) \cup (\varepsilon\omega \times \{1\})$.

The matrices $A^\varepsilon : \Omega \to \mathbb{R}^{N\times N}$ satisfy (1.3) again.

We consider the variational problem, with given data $F^\varepsilon \in \mathrm{L}^2(\Omega^\varepsilon)$, $G^\varepsilon \in \mathrm{L}^2(\Omega^\varepsilon)^N$, $h^\varepsilon \in \mathrm{H}^{-1/2}(0, 1)$,

$$\begin{aligned} &U^\varepsilon \in \mathcal{V}^\varepsilon = \left\{V \in \mathrm{H}^1\big(\Omega^\varepsilon\big),\ V = 0 \text{ on } \Gamma^\varepsilon\right\} \quad \text{and} \quad \forall V \in \mathcal{V}^\varepsilon, \\ &\int_{\Omega^\varepsilon} \left[A^\varepsilon\left(\frac{x'}{\varepsilon}, x_N\right)\nabla U^\varepsilon, \nabla V\right] \mathrm{d}x \\ &\quad = \int_{\Omega^\varepsilon} F^\varepsilon V \, \mathrm{d}x + \int_{\Omega^\varepsilon} \left[G^\varepsilon, \nabla V\right] \mathrm{d}x + \int_{\Sigma^\varepsilon} \varepsilon h^\varepsilon(x_N) V(x', x_N) \, \mathrm{d}\mathcal{H}_{N-2}(x') \, \mathrm{d}x_N. \end{aligned} \tag{2.13}$$

Again the notation of coefficients ($A^\varepsilon(\frac{x'}{\varepsilon}, x_N)$) is chosen for convenience. Classically (2.13) has a unique solution U^ε and U^ε solves (in a weak sense) a diffusion equation.

In this section, we write again $A^\varepsilon = (M^\varepsilon)^{-1} P^\varepsilon$, as in (1.5), but M^ε and P^ε are here given by

$$M^\varepsilon = \begin{pmatrix} R^\varepsilon & 0 \\ -L^\varepsilon R^\varepsilon & 1 \end{pmatrix}, \qquad P^\varepsilon = \begin{pmatrix} I & R^\varepsilon C^\varepsilon \\ 0 & a^\varepsilon_{NN} - L^\varepsilon R^\varepsilon C^\varepsilon \end{pmatrix}, \tag{2.14}$$

with $R^\varepsilon = ((A^\varepsilon)')^{-1}$.

THEOREM 2. – *Assume* (1.3) *and* (1.7)*, with* M^ε *and* P^ε *defined in* (1.5) *and* (2.14)*. Then the* H*-limit of* A^ε *is characterized by* $A = M^{-1}P$*, with* M *and* P *given by* (1.8)*, which we assume in addition. Define* f^ε *and* $g^\varepsilon = ((g^\varepsilon)', g_N^\varepsilon)$ *from* F *and* G *by* $f^\varepsilon(x) = F^\varepsilon(\varepsilon x', x_N)$*,* $g^\varepsilon(x) = G^\varepsilon(\varepsilon x', x_N)$*. Assume moreover that the data converge in the following sense*:

$$\begin{cases} f^\varepsilon \rightharpoonup f \quad \text{weakly in } \mathrm{L}^2(\Omega), \quad g^\varepsilon \rightharpoonup g \quad \text{weakly in } \mathrm{L}^2(\Omega)^N, \\ h^\varepsilon \rightharpoonup h \quad \text{weakly in } \mathrm{H}^{-1/2}(0,1) \end{cases} \tag{2.15}$$

and assume also that

$$\left\{ \mathrm{div}_{x'} \left(g^\varepsilon\right)' \right\}_\varepsilon \text{ is relatively compact in } \mathrm{H}^{-1}(\Omega). \tag{2.16}$$

Let $u^\varepsilon(x) = U^\varepsilon(\varepsilon x', x_N)$*, where* U^ε *is the solution of* (2.13) *and let* ∇^ε *be the operator defined by* $\nabla^\varepsilon v = (\frac{1}{\varepsilon}\nabla' v, \frac{\partial v}{\partial x_N}) = (\frac{1}{\varepsilon}\frac{\partial v}{\partial x_1}, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_{N-1}}, \frac{\partial v}{\partial x_N})$*. Then, when* ε *tends to zero,*

$$\begin{cases} u^\varepsilon \rightharpoonup u \quad \text{weakly in } \mathrm{H}^1(\Omega), \quad \frac{1}{\varepsilon}\nabla' u^\varepsilon \rightharpoonup \nabla' y \quad \text{weakly in } \mathrm{L}^2(\Omega)^N, \\ \sigma^\varepsilon = A^\varepsilon \nabla^\varepsilon u^\varepsilon \rightharpoonup \sigma = A \begin{pmatrix} \nabla' y \\ \partial u/\partial x_N \end{pmatrix} \quad \text{weakly in } \mathrm{L}^2(\Omega)^N, \end{cases} \tag{2.17}$$

where u *depends only on* x_N *and* (u, y) *is the unique solution of the limit variational problem*:

$$u \in \mathrm{H}_0^1(0,1),\ y \in \mathrm{L}^2\big(0,1; \mathrm{H}_m^1(\omega)\big) \quad \text{and} \quad \forall v \in \mathrm{H}_0^1(0,1),\ \forall z \in \mathrm{L}^2\big(0,1; \mathrm{H}_m^1(\omega)\big),$$

$$\int_\Omega \left[A \begin{pmatrix} \nabla' y \\ \partial u/\partial x_N \end{pmatrix}, \begin{pmatrix} \nabla' z \\ \partial v/\partial x_N \end{pmatrix} \right] \mathrm{d}x = \int_\Omega f v \,\mathrm{d}x + \int_\Omega \left[g, \begin{pmatrix} \nabla' z \\ \partial v/\partial x_N \end{pmatrix} \right] \mathrm{d}x + |\partial\omega| \int_0^1 h v \,\mathrm{d}x_N, \tag{2.18}$$

H_m^1 *denoting the subset of functions of* H^1*, having mean value zero.*

Remark 3. – In the case of fibered rods, $A^\varepsilon = A^\varepsilon(x')$, div $P^\varepsilon \equiv 0$ and (1.7) reads: $\{\mathrm{curl}_{x'} R^\varepsilon\}_\varepsilon$ is relatively compact in $\mathrm{H}^{-1}(\Omega)^{(N-1)\times(N-1)\times(N-1)}$ and $\{\mathrm{curl}_{x'} L^\varepsilon R^\varepsilon\}_\varepsilon$ is relatively compact in $\mathrm{H}^{-1}(\Omega)^{(N-1)\times(N-1)}$. Examples are given in [12].

It is possible to express the limit problem in terms of u only, by eliminating y. This allows us to compare our result with the more general, but not explicit one, of F. Murat [18].

Detailed proofs and some complements are given in [12].

References

[1] A. Bensoussan, J.L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, Stud. Math. Appl., Vol. 5, North-Holland, 1978.

[2] D. Caillerie, Homogénéisation des équations de la diffusion stationnaire dans les domaines cylindriques aplatis, RAIRO Analyse Numérique 15 (1981) 295–319.

[3] D. Caillerie, Thin elastic and periodic plates, Math. Methods Appl. Sci. 6 (1984) 159–191.

[4] P.G. Ciarlet, Plates and Junctions in Elastic Multistructures: An Asymptotic Analysis, Masson, Paris, 1990.

[5] P.G. Ciarlet, Mathematical Elasticity, Vol. II: Theory of Plates, North-Holland, Amsterdam, 1997.

[6] P.G. Ciarlet, P. Destuynder, A justification of the two-dimensional linear plate model, J. Mécanique 18 (1979) 315–344.

[7] P. Courilleau, Homogénéisation et compacité par compensation, C. R. Acad. Sci. Paris, Série I 332 (2001) 991–994.

[8] P. Courilleau, S. Fabre, J. Mossino, Homogénéisation de problèmes non linéaires dans des milieux quasi-stratifiés, C. R. Acad. Sci. Paris, Série I 328 (1999) 401–406.

[9] P. Courilleau, S. Fabre, J. Mossino, Homogenization of some nonlinear problems with specific dependence upon coordinates, Boll. Un. Mat. Ital. B (8) 4 (2001) 711–729.

[10] A. Damlamian, M. Vogelius, Homogenization limits of diffusion equations in thin domains, Math. Modelling Numer. Anal. 22 (1) (1988) 53–74.

[11] S. Fabre, J. Mossino, H-convergence of multiplicable matrices, Calc. Var. 7 (1998) 125–139.

[12] B. Gustafsson, J. Mossino, Nonperiodic explicit homogenization and reduction of dimension: the linear case, detailed version, in preparation.

[13] R. Kohn, M. Vogelius, A new model for thin plates with rapidly varying thickness II: a convergence proof, Quart. Appl. Math. 43 (1985) 1–22.
[14] H. Le Dret, Problèmes variationnels dans les multidomaines : modélisation des jonctions et applications, Masson, Paris, 1991.
[15] H. Le Dret, A. Raoult, The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity, J. Math. Pures Appl. 74 (1995) 549–578.
[16] H. Le Dret, A. Raoult, The membrane shell model in nonlinear elasticity, J. Nonlinear Sci. 6 (1996) 58–84.
[17] F. Murat, H-convergence, Séminaire d'Analyse Fonctionnelle et Numérique, University of Alger, 1977–1978.
[18] F. Murat, Private communication to J. Mossino.
[19] F. Murat, A. Sili, Problèmes monotones dans des cylindres de faible diamètre formés de matériaux hétérogènes, C. R. Acad. Sci. Paris, Série I 320 (1995) 1199–1204.
[20] A. Sili, Homogénéisation dans des cylindres minces, C. R. Acad. Sci. Paris, Série I 332 (2001) 777–782.
[21] A. Sili, Homogenization of the heat equation, to appear.
[22] L. Tartar, Homogénéisation et compacité par compensation, Cours Peccot, Collège de France, partially written in [17].