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Algebraic braided model of the affine line and difference calculus on a topological space

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Abstract In a previous Note [1], we suggested a quantum model of the unit interval $[0,1]$, using convergent power series, parametrized by a variable q (a remarkable example is the quantum exponential, defined by Euler). In the present Note, we suggest a simpler model based on functions $f = f(x) : \mathbf{Z} \to k$ (with an arbitrary commutative ring k) which are constant when $x \mapsto +\infty$ or $x \mapsto -\infty$ and their "differentials" considered as functions $x \mapsto f(x+1) - f(x)$ (difference calculus). Thanks to this new "*differential calculus over the integers*", we can associate to any simplicial set or topological space X a braided differential graded algebra $\mathcal{D}^*(X)$ which is similar in spirit to the algebra $\mathcal{W}^*(X)$ introduced in [1]. We notice that the p-homotopy type of X can be read from the braiding of $\mathcal{D}^*(X)$. In particular, if $k = \mathbf{Z}$, we recover in a purely algebraic way the integral cohomology, Steenrod operations, homotopy groups from this braiding. ***To cite this article: M. Karoubi, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 121–126.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Modèle algébrique tressé de la droite affine et calcul aux différences sur un espace topologique

Résumé Dans une Note précédente [1], nous avons proposé un modèle quantique de l'intervalle $[0,1]$, à partir de séries convergentes dépendant d'un paramètre q (un exemple notable étant l'exponentielle quantique, dûe à Euler). Dans cette Note, nous suggérons un modèle plus simple construit à partir de fonctions $f = f(x) : \mathbf{Z} \to k$ (k étant un anneau commutatif quelconque) constantes quand $x \mapsto +\infty$ ou $x \mapsto -\infty$ et leurs « différentielles » $\mathrm{d}f$ que nous interpretons comme les fonctions $x \mapsto f(x+1) - f(x)$ (calcul aux différences). Grâce à ce nouveau « *calcul différentiel sur les nombres entiers* », nous pouvons associer à un ensemble simplicial ou espace topologique quelconque X une algèbre différentielle graduée tressée $\mathcal{D}^*(X)$, analogue en esprit à l'algèbre $\mathcal{W}^*(X)$ considérée dans [1]. Il convient de remarquer que le p-type d'homotopie de l'ensemble simplicial X « se lit » essentiellement sur le tressage de l'algèbre $\mathcal{D}^*(X)$. En particulier, si $k = \mathbf{Z}$, nous retrouvons de manière purement algébrique la cohomologie entière, les opérations de Steenrod, les groupes d'homotopie à partir de ce tressage. ***Pour citer cet article : M. Karoubi, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 121–126.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Version française abrégée

1. Calcul aux différences sur la droite affine

1.1. Soit A la k-algèbre des fonctions $f = f(x) : \mathbf{Z} \to k$ qui sont constantes quand x tend vers $+\infty$ ou $-\infty$ (deux limites indépendantes). On peut munir A de l'automorphisme $\alpha : A \to A$ défini par $\alpha(f)(x) = f(x+1)$. Pour alléger les notations, on pose $\bar{f} = \alpha(f)$. Nous définissons alors $\Omega^1(A)$ comme l'idéal *à gauche* de A formé des fonctions f qui tendent vers 0 quand x tend vers $\pm\infty$. Cet idéal est vu comme module *à droite* en posant $\omega \cdot f = \bar{f} \cdot \omega$ pour $f \in A$ et $\omega \in \Omega^1(A)$. Nous définissons enfin la « différentielle non commutative »

$$d : A \to \Omega^1(A)$$

par la formule $\mathrm{d}(f)(x) = f(x+1) - f(x)$ (calcul aux différences). Cette différentielle vérifie la formule de Leibniz $\mathrm{d}(f \cdot g) = \mathrm{d}f \cdot g + f \cdot \mathrm{d}g = \bar{g} \cdot \mathrm{d}f + f \cdot \mathrm{d}g$, comme il se doit.

1.2. *Notations.* Pour souligner la variable x, nous poserons $A = \mathcal{D}^0(x)$ and $\Omega^1(A) = \mathcal{D}^1(x)$. Avec ces notations, le théorème suivant est évident :

THÉORÈME 1.3 (Lemme de Poincaré pour la droite affine). – *La différentielle*

$$\mathrm{d} : \mathcal{D}^0(x) \to \mathcal{D}^1(x)$$

est surjective et son noyau se réduit aux fonctions constantes.

1.4. En suivant [2], nous définissons un tressage R sur l'algèbre différentielle graduée

$$\mathcal{D}^*(x) = \mathcal{D}^0(x) \oplus \mathcal{D}^1(x)$$

par les formules que voici (où f et g sont de degré 0, ω et θ de degré 1) :

$$\begin{aligned}
R(f \otimes g) &= g \otimes f,\\
R(\omega \otimes g) &= \bar{g} \otimes \omega,\\
R(f \otimes \mathrm{d}g) &= \mathrm{d}g \otimes f + (g - \bar{g}) \otimes \mathrm{d}f,\\
R(\omega \otimes \theta) &= -\bar{\theta} \otimes \omega
\end{aligned}$$

[noter que l'automorphisme $\theta \mapsto \bar{\theta}$ de $\Omega^1(A)$ est induit par l'automorphisme α de A].

1.5. Il est important de remarquer que l'algèbre $\mathcal{D}^*(x)$ possède deux augmentations naturelles obtenues en faisant tendre la variable x vers $+\infty$ ou $-\infty$ respectivement. Celles-ci sont compatibles avec le tressage, ainsi qu'il a été explicité dans [1], p. 760, pour l'algèbre $\mathcal{W}^*(x)$.

2. Calcul aux différences sur un ensemble simplicial et un espace topologique

2.1. Si $(x_0, \ldots, x_r)$ sont $r+1$ indéterminées, $\mathcal{D}^*(x_0, \ldots, x_r)$ désigne le produit tensoriel gradué $\mathcal{D}^*(x_0) \otimes \cdots \otimes \mathcal{D}^*(x_r)$. Si Δ_r est le r-simplexe standard, on définit $\mathcal{D}^*(\Delta_r)$ comme l'égalisateur des deux flèches

$$\prod_i \mathcal{D}^*\big(x_0, \ldots, \hat{x}_1, \ldots, x_r\big) \rightrightarrows \prod_{1<j} \mathcal{D}^*(x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_r)$$

obtenues en posant x_i ou x_j égal à $-\infty$.

THÉORÈME 2.2. – *Les* $\mathcal{D}^*(\Delta_r)$, $r = 0, 1, \ldots,$ *définissent une algèbre différentielle graduée simpliciale avec des opérateurs face obtenues en faisant* $x_i = +\infty$. *Pour tout* r, *sa cohomologie est concentrée en* *degré* 0 *et est isomorphe à* k (*lemme de Poincaré pour le* r*-simplexe*). *De plus, les « tressages » obtenues par produits tensoriels adéquats de* R*, soit*

$$\mathcal{D}^*(x_0,\ldots,x_r)\otimes\mathcal{D}^*(x_0,\ldots,x_s)\longrightarrow\mathcal{D}^*(x_0,\ldots,x_s)\otimes\mathcal{D}^*(x_0,\ldots,x_r)$$

induisent un « tressage bisimplicial »

$$\mathcal{D}^*(\Delta_r)\otimes\mathcal{D}^*(\Delta_s)\longrightarrow\mathcal{D}^*(\Delta_s)\otimes\mathcal{D}^*(\Delta_r).$$

2.3. En suivant [1], nous définissons $\mathcal{D}^*(X)$ pour un ensemble simplicial X quelconque comme le « produit réduit »

$$\mathcal{D}^*(X)=C^\sharp(X)\nabla\mathcal{D}^*(\Delta_\sharp),$$

où $C^\sharp(X)$ est le k-module cosimplicial des cochaînes usuelles sur X.

THÉORÈME 2.4. – *La structure tressée bisimpliciale sur les* $\mathcal{D}^*(\Delta_r)$ *induit un tressage sur l'algèbre* $\mathcal{D}^*(X)$. *Sa cohomologie est naturellement isomorphe à la cohomologie de* X *à coefficients dans* k*, munie de sa structure multiplicative usuelle.*

2.5. Plus généralement, si $\mathcal{F}$ est un faisceau en k-algèbres commutatives sur un espace topologique quelconque X et si $\mathcal{F}^\sharp$ est la résolution flasque cosimpliciale de Godement du faisceau $\mathcal{F}$, $\mathcal{F}^\sharp\nabla\mathcal{D}^*(\Delta_\sharp)$ définit un faisceau en ADG tressées, dont les sections calculent aussi la cohomologie à coefficients dans $\mathcal{F}$.

2.6. Nous pouvons enfin définir une version « stabilisée » $\widehat{\mathcal{D}}^*(X)$ de $\mathcal{D}^*(X)$, en remplaçant $\mathcal{D}^*(\Delta_r)$ par $\mathrm{colim}_n\,\mathcal{D}^*(\Delta_r)^{\otimes n}$. L'ADG tressée obtenue est alors « spéciale » dans le sens que le noyau symétrique de $\widehat{\mathcal{D}}^*(X)^{\otimes n}$ est quasi-isomorphe à $\widehat{\mathcal{D}}^*(X)^{\otimes n}$ (*cf.* [1,2] pour les définitions).

Le théorème suivant se déduit des considérations de [2,3] et [4] :

THÉORÈME 2.7. – *Soient* X *et* Y *deux ensembles simpliciaux connexes nilpotents et de type fini.*[1] *On suppose qu'il existe une suite en zigzag de quasi-isomorphismes d'ADG tressées spéciales* (*avec* $k=\mathbf{Z}$)

$$\widehat{\mathcal{D}}^*(X)\longrightarrow A\longleftarrow B\longrightarrow\cdots\longleftarrow\widehat{\mathcal{D}}^*(Y).$$

Alors X *et* Y *ont le même type d'homotopie rationnel et le même type d'homotopie* p*-adique pour tout nombre premier* p.

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1. Difference calculus on the affine line

1.1. Let us call A the k-algebra of functions $f=f(x):\mathbf{Z}\to k$ which are constant when x goes to $+\infty$ or $-\infty$ (two independent limits). It is easy to show that any element of A can be written in a unique way

$$f=\lambda\cdot 1+\mu\cdot Y+g,$$

where 1 is the unit function, Y the "Heaviside function" defined by $Y(x)=1$ if $x\geqslant 0$, 0 otherwise and g a finite linear combination of "Dirac functions" (i.e., characterictic functions of points in $\mathbf{Z}$). In the same way, $A\otimes A$ may be identified with the algebra of functions of 2 variables $h(x,y):\mathbf{Z}\times\mathbf{Z}\to k$ which can be written (uniquely) in the form

$$\lambda\cdot(1\otimes 1)+\mu\cdot(1\otimes Y)+\nu\cdot(Y\otimes 1)+\pi\cdot(Y\otimes Y)+h(x,y),$$

where h is a finite linear combination of Dirac functions on $\mathbf{Z}\times\mathbf{Z}$. The A-module of noncommutative differential forms of degree one, denoted by $\Omega^1_{\mathrm{nc}}(A)$, is the kernel of the multiplication map

$$m:A\otimes A\longrightarrow A$$

which can be interpreted by putting $x = y$ in the function $h(x, y)$. This implies that $\lambda = 0$ and $\mu + \nu + \pi = 0$ together with $h(x, x) = 0$. Since the non commutative differential $d_{\mathrm{nc}} : A \to \Omega^1_{\mathrm{nc}}(A)$ sends g to the function $h(x, y) = g(y) - g(x)$, we see that any element of $\Omega^1_{\mathrm{nc}}(A)$ is a finite linear combination of $\mathrm{d}Y$, $Y \cdot \mathrm{d}Y$ and $f_i \cdot \mathrm{d}g_i$ where f_i and g_i are Dirac functions.

1.2. Let us now consider the automorphism α of A defined by $\alpha(f)(x) = f(x + 1)$. In order to shorten the notations, we shall often write $\bar{f}$ instead of $\alpha(f)$. Following [2], we define $\Omega^1_\alpha(A)$-denoted simply $\Omega^1(A)$ – as the quotient of $\Omega^1_{\mathrm{nc}}(A)$ by relations of the type $\mathrm{d}g \cdot f = \alpha(f) \cdot \mathrm{d}g$. In terms of functions of 2 variables $h(x, y)$ belonging to $A \otimes A$, we have therefore the relation

$$h(x, y) \cdot f(y) = f(x + 1)h(x, y).$$

This means that we should put $y = x + 1$ in the expression of h (take for h a Dirac function). In other words, the quotient map $\Omega^1_{\mathrm{nc}}(A) \to \Omega^1(A) \subset A$ sends $h(x, y)$ to the function of *one* variable $h(x, x + 1)$. Therefore, the composition of $d_{\mathrm{nc}} : A \to \Omega^1_{\mathrm{nc}}(A)$ with the canonical map $\Omega^1_{\mathrm{nc}}(A) \to \Omega^1(A) \subset A$ sends f to the function $h(x) = f(x + 1) - f(x)$ (difference calculus). Of course, we have the usual Leibniz rule $\mathrm{d}(f \cdot g) = \mathrm{d}f \cdot g + f \cdot \mathrm{d}g = \bar{g} \cdot \mathrm{d}f + f \cdot \mathrm{d}g$, where the left A-module structure on $\Omega^1(A)$ is the usual multiplication of functions. We notice that any element of $\Omega^1(A)$ is a finite linear combination of elements $\mathrm{d}y$ and $f_i \cdot \mathrm{d}g_i$ where f_i and g_i are Dirac functions. Therefore, $\Omega^1(A)$ may be identified with the left ideal in A, with functions going to 0 at both infinities.

1.3. *Notation.* In order to emphasize the variable x, we shall often write $A = \mathcal{D}^0(x)$ and $\Omega^1(A) = \mathcal{D}^1(x)$.

THEOREM 1.4 (Poincaré's lemma for the affine line). – *The differential*

$$\mathrm{d} : \mathcal{D}^0(x) \longrightarrow \mathcal{D}^1(x)$$

is surjective and its kernel consists of constant functions.

Proof. – If $f \in \mathrm{Ker}(d)$, we have $f(x) = f(x + 1)$ and therefore f is constant. On the other hand, if $\omega(x) \in \Omega^1(A) \subset A$, we can define its "antiderivative" $f(x)$ as $\sum_{t=-\infty}^{x-1} \omega(t)$ (which is in fact a finite sum). It is clear that $f(x + 1) - f(x) = \omega(x)$.

1.5. Following [2], we define a braiding R on the differential graded algebra

$$\mathcal{D}^*(x) = \mathcal{D}^0(x) \oplus \mathcal{D}^1(x)$$

by the following formulas (where f and g are of degree 0, ω and θ of degree 1)

$$R(f \otimes g) = g \otimes f,$$
$$R(\omega \otimes g) = \bar{g} \otimes \omega,$$
$$R(f \otimes \mathrm{d}g) = \mathrm{d}g \otimes f + (g - \bar{g}) \otimes \mathrm{d}f,$$
$$R(\omega \otimes \theta) = -\bar{\theta} \otimes \omega.$$

[the automorphism $\theta \mapsto \bar{\theta}$ is induced by the automorphism α of A].

Checking the braid relations is an easy (but tedious) exercise.

1.6. Finally and very importantly, we notice that the algebra $\mathcal{D}^*(x)$ has *two* natural augmentations by letting the variable x go to $+\infty$ or $-\infty$ respectively. These augmentations are compatible with the braiding, as for the algebra $\mathcal{W}^*(x)$ considered in [1], p. 760.

2. Difference calculus on a simplicial set and a topological space

2.1. We follow now the pattern described in [1]. If $(x_0, \ldots, x_r)$ are $(r+1)$ indeterminates, $\mathcal{D}^*(x_0, \ldots, x_r)$ denotes the graded tensor product $\mathcal{D}^*(x_0) \otimes \cdots \otimes \mathcal{D}^*(x_r)$. If Δ_r is the standard r-simplex, we define

$\mathcal{D}^*(\Delta_r)$ as the kernel of the difference of the morphisms

$$\prod_i \mathcal{D}^*(x_0,\ldots,\hat{x}_i,\ldots,x_r) \rightrightarrows \prod_{i<j} \mathcal{D}^*(x_0,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_r)$$

obtained by putting the variable x_i or x_j equal to $-\infty$.

THEOREM 2.2. – *The* $\mathcal{D}^*(\Delta_r)$ *define a* simplicial *differential graded algebra with the face maps induced by the evaluations* $x_i = +\infty$. *For each* r, *its cohomology is concentrated in degree 0 and is isomorphic to* k (*Poincaré's lemma for the* r*-simplex*)*. Moreover, the obvious "braiding" obtained by suitable tensor products of the braiding R defined in* 1.5:

$$\mathcal{D}^*(x_0,\ldots,x_r)\otimes\mathcal{D}^*(x_0,\ldots,x_s) \longrightarrow \mathcal{D}^*(x_0,\ldots,x_s)\otimes\mathcal{D}^*(x_0,\ldots,x_r)$$

induces a bisimplicial *braiding*

$$\mathcal{D}^*(\Delta_r)\otimes\mathcal{D}^*(\Delta_s) \longrightarrow \mathcal{D}^*(\Delta_s)\otimes\mathcal{D}^*(\Delta_r).$$

2.3. We define $\mathcal{D}^*(X)$ for any simplicial set X as the "reduced product"

$$\mathcal{D}^*(X) = C^\sharp(X)\nabla\mathcal{D}^*(\Delta_\sharp),$$

where $C^\sharp(X)$ is the *cosimplicial* k-module of usual cochains on X. As defined by Bousfield and Kan, the reduced product of a cosimplicial module $C^\sharp$ and a semi-simplicial module $S_\sharp$ is the quotient of the direct sum $C^n\otimes S_n$ by the usual identifications ([1], p. 760).

THEOREM 2.4. – *The bisimplicial braided DGA structure on the* $\mathcal{D}^*(\Delta_r)$ *induces a* globally *defined braided DGA structure on* $\mathcal{D}^*(X)$. *Its cohomology is naturally isomorphic to the cohomology of* X *with coefficients in* k, *provided with its usual multiplicative structure.*

2.5. We notice an important advantage of the differential graded algebra $\mathcal{D}^*(X)$, compared to the usual algebra of cochains $C^*(X)$. If X and Y are two spaces, we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^*(X)\otimes\mathcal{D}^*(Y) & \longrightarrow & \mathcal{D}^*(X\times Y)\\
\downarrow R_{X,Y} & & \downarrow \sigma\\
\mathcal{D}^*(Y)\otimes\mathcal{D}^*(X) & \longrightarrow & \mathcal{D}^*(Y\times X)
\end{array}$$

where σ is induced by the permutation of the factors X and Y and where $R_{X,Y}$ is the "braiding" (generalized easily to a couple of spaces X and Y; not just for $X=Y$).

2.6. More generally, if $\mathcal{F}$ is a sheaf of commutative k-algebras on a topological space X and if $\mathcal{F}^\sharp$ is the canonical cosimplicial Godement flabby resolution of $\mathcal{F}$, the $\mathcal{F}^r\nabla\mathcal{D}^*(\Delta_r)$ define a braided differential graded sheaf which space of sections computes also the cohomology of X with values in $\mathcal{F}$. Therefore, we can apply this type of difference calculus to *any* topological space.

2.7. As in [1], we can define a "stabilized" version of $\mathcal{D}^*(X)$, replacing $\mathcal{D}^*(\Delta_r)$ by $\operatorname{colim}_n \mathcal{D}^*(\Delta_r)^{\otimes n} = \widehat{\mathcal{D}}^*(\Delta_r)$. We write $\widehat{\mathcal{D}}^*(X)$ for the corresponding braided differential graded algebra $C^\sharp(X)\nabla\widehat{\mathcal{D}}^*(\Delta_\sharp)$. It is a *special* braided DGA with the definition given in [2]: the symmetric kernel of $\widehat{\mathcal{D}}^*(X)^{\otimes n}$ is quasi-isomorphic to $\widehat{\mathcal{D}}^*(X)^{\otimes n}$.

The following theorem is an application of the previous considerations. Its proof is based on the main results of [2,3] and [4]:

THEOREM 2.8. – *Let* X *and* Y *be two connected simplicial sets, nilpotent and of finite type.*[2] *Let us assume there exists a zigzag sequence of quasi-isomorphisms of special braided DGA's* (*with* $k=\mathbf{Z}$)

$$\widehat{\mathcal{D}}^*(X) \longrightarrow A \longleftarrow B \longrightarrow \cdots \longleftarrow \widehat{\mathcal{D}}^*(Y).$$

Then X and Y have the same rational homotopy type and the same p-adic homotopy type for all p.

2.9. We conjecture that this theorem is still true without the word “special” and with $\mathcal{D}$ instead of $\widehat{\mathcal{D}}$. We also conjecture that X and Y have the same homotopy type (not just mod p and rationnally). In support of the first conjecture, we notice that the cohomology mod p of the iterated loop space $\mathcal{L}^r(X)$ of the base space X can be computed from a suitable *r-iterated bar-construction* of the braided differential graded algebra $A = \mathcal{D}^*(X)$, although A is not commutative. More precisely, we can define a r-simplicial DGA by putting [3]

$$(i_1, \ldots, i_r) \mapsto A^{\otimes i_1, \ldots, i_r}.$$

We use the braiding to define the face maps and to check the commutativity of the obvious diagrams. As an illustration, the following picture in a braided DGA is a substitute (up to sign) for the identity $(ab)(cd) = (ac)(bd)$ in a commutative DGA. It shows for instance that the multiplication $A \otimes A \to A$ is a ring map for a suitable ring structure on $A \otimes A$ deduced from the braining.

THEOREM 2.10. – *Let X be a connected simplicial set, nilpotent and of finite p-type. Then the cohomology with coefficients in $k = \mathbf{Z}/p$ of the r-iterated loop space $\mathcal{L}^r(X)$ is the cohomology of the total complex associated to the r-simplicial DGA*

$$(i_1, \ldots, i_r) \mapsto A^{\otimes i_1, \ldots, i_r}$$

considered above, with $A = \mathcal{D}^*(X)$.

Remark 2.11. – This theorem provides a new algebraic strategy to compute homotopy groups via homological algebra techniques (compare with [5]).

[1] Ceci signifie que la tour de Postnikov est choisie en sorte que chaque fibre est de type $K(G, n)$, où G est un groupe abélien de type fini.

[2] This means that its Postnikov tower is choosen such that each fiber is of the type $K(G, n)$, where G is a finitely generated Abelian group.

[3] More precisely, one has to take the quotient of $A^{\otimes n}$ by the action of the pure braid group P_n.

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