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Hasse–Witt invariants of symmetric complexes: an example from geometry

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Abstract Jardine has defined Hasse–Witt invariants for symmetric bundles over schemes. This definition can be extended to symmetric complexes, that is symmetric objects in the derived category of bounded complexes of vector bundles over a scheme. In this Note we show how one can use these generalized invariants to give a neater proof of a comparison result on Hasse–Witt invariants of symmetric bundles attached to tame coverings of schemes. ***To cite this article: P. Cassou-Noguès et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 839–842.***
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Invariants de Hasse–Witt de complexes symétriques : un exemple géométrique

Résumé Jardine a défini des invariants de Hasse–Witt pour des fibrés symétriques sur des schémas. On peut étendre cette définition aux complexes symétriques, c'est-à-dire aux objets symétriques de la catégorie dérivée des complexes bornés des fibrés vectoriels sur un schéma. Dans cette Note nous montrons comment on peut utiliser ces invariants généralisés pour obtenir une démonstation plus directe d'un résultat de comparaison pour les invariants de Hasse–Witt de fibrés symétriques attachés à des revêtements de schémas modérés. ***Pour citer cet article : P. Cassou-Noguès et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 839–842.***
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1. Introduction

In a recent paper we obtained a generalisation of formulae of Serre and Esnault–Kahn–Viehweg which describe the difference between two kinds of characteristic classes attached to a tame covering of schemes with odd ramification as a sum of various local terms (*see* [9,10,4] and [3]). One of the main steps in the proof of our formulae is a result in which we compare the Hasse–Witt invariants of two symmetric bundles which coincide on the generic fibre (*see* [3], Theorem 0.1). The Hasse–Witt invariants we use in [3] are those defined by Jardine in [5]. The aim of this Note is to give a neater presentation of that comparison result by using ideas which originated in the work of Balmer on "triangular" Witt groups [1,2]. More precisely, we

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interpret the comparison result as an equality between Hasse–Witt invariants of symmetric complexes. The observation that one can define such generalized invariants is due to Saito (*see* [8]). Their existence also follows from the work in [11].

In the first part of this Note we indicate how to define Hasse–Witt invariants of symmetric complexes. In the second part we begin by recalling the set-up of [3], and then we state and prove our main result.

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2. Hasse–Witt invariants of symmetric complexes

A *symmetric bundle* (E,ϕ) over a noetherian $\mathbf{Z}\left[\frac{1}{2}\right]$-scheme Y is a vector bundle E over Y equipped with a symmetric isomorphism ϕ between E and its Y-dual E^D, that is to say $\phi : E \cong E^D$ and ϕ is equal to its transpose ϕ^D after identifying E with E^{DD} (*see*, e.g., [3], 1.c). A *symmetric complex* on Y is a symmetric object $(P_\bullet,\phi)$ of the derived category $\mathcal{D}^b(Y)$ of bounded complexes of Y-vector bundles. This is a triangulated category with a duality which extends D: namely the localisation of the functor which sends a complex $P_\bullet$ to the dual complex $P_\bullet^D$. Symmetry is defined using the natural identification of a complex with its double dual (*see* Section 2 of [1]). A symmetric bundle may therefore be viewed as a symmetric complex concentrated in degree zero. Following Knebusch, we say that a symmetric bundle (E,ϕ) is *metabolic* if it contains a Lagrangian, that is to say a totally isotropic sub-bundle with Y-rank equal to half the rank of E. Metabolic bundles are trivial in the Witt group $W(Y)$, but the converse does not in general hold (*see* [7], Example 2.10 for an example).

More generally one can define a metabolic object in $\mathcal{D}^b(Y)$: we say that $(P_\bullet,\phi)$ is metabolic with Lagrangian $L_\bullet$ if there is a distinguished triangle $L_\bullet \xrightarrow{i} P_\bullet \xrightarrow{i^D\circ\phi} L_\bullet^D \xrightarrow{w} TL_\bullet$ in the derived category with the duality condition that $T(w^D)=w$. One can then define a Witt group for $\mathcal{D}^b(Y)$, and an object in $\mathcal{D}^b(Y)$ is metabolic if and only if it is zero in this Witt group (*see* Theorem 3.5 in [1]). We state without proof the following result (*see* [8] or [11] for details).

PROPOSITION 1. – *For any symmetric complex $(P_\bullet,\phi)$ there exists a symmetric bundle (E',γ) such that the orthogonal sum $(P_\bullet,\phi)\perp(E',\gamma)$ is metabolic with Lagrangian given by $P_{<0}\oplus E'$.*

We are now in a position to define the Hasse–Witt invariant of a symmetric complex. Recall that the total Hasse–Witt invariant of a symmetric bundle $E=(E,\phi)$ over Y is

$$w_t(E)=\sum_{i\geqslant 0} w_i(E)t^i,$$

where $w_i(E)$ belongs to $H^i(Y):=H^i(Y_{et},\mathbf{Z}/2\mathbf{Z})$ (*see*, e.g., 1.e in [3]). This invariant does not in general vanish on metabolic bundles. If for example E is a metabolic bundle over Y with Lagrangian V of rank n and if $c_i(V)$ is the i-th Chern class in $H^{2i}(Y)$, then we have the following result.

LEMMA 2 (Proposition 5.5 in [4]). – *With the above notation*

$$w_t(E)=d_t(V)\overset{\text{defn}}{=}\sum_{i=0}^{n}\bigl(1+(-1)t\bigr)^{n-i}c_i(V)t^{2i}.$$

We extend the definition of $d_t(-)$ to complexes by multiplicativity, so that $d_t(L_\bullet)=\prod_i d_t(L_i)^{(-1)^i}$. Guided by the lemma, forcing additivity and using the notation of the proposition, we then recoup Saito's definition of Hasse–Witt classes of [8] for symmetric complexes by putting

$$w_t(P_\bullet,\phi)\overset{\text{defn}}{=}w_t\bigl(E',-\gamma\bigr)d_t(P_{<0}).$$

We note for future reference that these Hasse–Witt classes have all the standard properties of characteristic classes. In particular they are natural with respect to pullback and satisfy the Whitney sum formula on sums of symmetric complexes; furthermore Lemma 2 extends to metabolic complexes (*see* [8]).

3. Tame coverings of schemes

In what follows all schemes will be noetherian and defined over $S = \mathrm{Spec}(\mathbf{Z}[\frac{1}{2}])$. Let $\widetilde{X}$ be a connected, projective, regular scheme which is either defined over $\mathrm{Spec}(\mathbf{F}_p)$ or is flat over S and which supports an action by a finite group G, so that the quotient $\widetilde{\pi} : \widetilde{X} \to Y := \widetilde{X}/G$ exists. Assume furthermore that Y is regular and that $\widetilde{\pi}$ is *tame*; that is to say that $\widetilde{\pi}$ is a torsor outside of a divisor b of Y with normal crossings along which the ramification is tame. Moreover we suppose that one (and hence all) Sylow 2-groups G_2 act freely on $\widetilde{X}$. Let H be a subgroup of G and let $X := \widetilde{X}/H$, which we also assume to be regular. We shall work with the cover $\pi : X \to Y$. By assumption the ramification index $e(\xi_h)$ of a codimension one point ξ_h in X above the generic point η_h of the irreducible component b_h of b is necessarily odd. Thus we can define the sheaf

$$\mathcal{D}_{X/Y}^{-1/2} = O_X\left(\sum_{h,\xi_h \to \eta_h} \left(\frac{e(\xi_h)-1}{2}\right)\overline{\{\eta_h\}}\right).$$

Together with the trace form this defines a symmetric bundle $(\pi_*\mathcal{D}_{X/Y}^{-1/2}, \mathrm{Tr}_{X/Y})$ over Y. Our aim is to re-prove part of the following result, which reduces the computation of the Hasse–Witt invariants of this bundle to the case where π is étale (*see* [3], Theorem 0.1).

THEOREM 3. – *Let $Z = \widetilde{X}/G_2$, $T' = Z \times_Y X$ and let T denote the normalisation of T'. Then*

(i) *T is regular and $\pi_Z : T \to Z$ is étale;*

(ii) *$\phi : Z \to Y$ induces an injection $\phi^* : H^*(Y) \to H^*(Z)$;*

(iii) *there is a sequence of O_Z-locally free sheaves $\mathcal{G}^{(h)}$ numbered by the irreducible components $\{b_h\}_{h=1}^m$ of the branch locus b, such that*

$$w_t\left(\phi^*\left(\pi_*\mathcal{D}_{X/Y}^{-1/2}, \mathrm{Tr}_{X/Y}\right)\right) = w_t\left(\pi_* O_T, (-1)^m \mathrm{Tr}_{T/Z}\right)^{(-1)^m} \prod_{h=0}^{m-1} d_t\left(\mathcal{G}^{(h)}\right)^{(-1)^h}.$$

Remark. – (a) By (i) we know that the first term on the right-hand side of the formula in (iii) can be determined using the results of [4] and [6].

(b) The above formula can be made completely explicit in degrees 1 and 2 (*see* Theorem 0.2 in [3]).

(c) One point that we wish to stress in this Note is that the above formula becomes entirely natural when viewed as an equality of invariants of complexes.

Proof. – For (i) and (ii) we refer to [3]. The assumptions on the ramification are heavily used to prove (i). In Section 3 of [3] we show how to decompose the normalisation map $T \to T'$ into a sequence of flat Z-covers $T = T^{(m)} \to \cdots \to T^{(0)} = T'$ with $\pi_h : T^{(h)} \to Z$ having the property that $\mathcal{D}_{T^{(h)}/Z}^{-1/2}$ is a well-defined locally free $T^{(h)}$-sheaf. We then put $\Lambda^{(h)} := \pi_{h*}\mathcal{D}_{T^{(h)}/Z}^{-1/2}$ and define $I^{(h)} = \Lambda^{(h)} \cap \Lambda^{(h+1)}$ and $\mathcal{G}^{(h)} = (\Lambda^{(h)} + \Lambda^{(h+1)})/I^{(h)}$ so that we have the basic exact sequences

$$0 \to I^{(h)} \to \Lambda^{(h)} \oplus \Lambda^{(h+1)} \to \mathcal{G}^{(h)} \to 0. \tag{1}$$

Note that $\Lambda^{(0)}$ and $\Lambda^{(m)}$, when endowed with the trace form, are precisely the symmetric bundles that we wish to compare in (iii). The $\Lambda^{(h)}$ all coincide on the generic fibre and in 3.12 of [3] we show that the $I^{(h)}$ and $\mathcal{G}^{(h)}$ are all locally free over Z. Furthermore for $1 \leqslant h \leqslant m$ the sum of $(\Lambda^{(h)}, (-1)^h\,\mathrm{Tr})$

and $(\Lambda^{(h+1)}, (-1)^{h+1}\,\mathrm{Tr})$ is metabolic with Lagrangian $I^{(h)} = \mathcal{G}^{(h)D}$ (*see* loc. cit.). From Theorem 3.5 and Example 3.8 in [1], we deduce that $(\Lambda^{(0)}, \mathrm{Tr})$ is Witt equivalent to $(\Lambda^{(m)}, \mathrm{Tr})$ and that $(\Lambda^{(0)}, \mathrm{Tr}) \perp (\Lambda^{(m)}, -\mathrm{Tr})$ is metabolic in $\mathcal{D}^b(Z)$ with Lagrangian complex $M_\bullet$ where $M_i = \{0\}$ if $i \notin \{0,1\}$ and where $M_0 = \bigoplus_{h=0}^{m-1} I^{(h)}$ and $M_1 = \bigoplus_{h=1}^{m-1} \Lambda^{(h)}$. Hence from the additivity property of the Hasse–Witt invariant and Lemma 2, all extended to complexes, we deduce that

$$w_t\big(\Lambda^{(0)}, \mathrm{Tr}\big) w_t\big(\Lambda^{(m)}, -\mathrm{Tr}\big) = d_t(M_0 - M_1) := d_t\left(\sum_{h=0}^{m-1} I^{(h)} - \sum_{h=1}^{m-1} \Lambda^{(h)}\right).$$

Again by Lemma 2 we get $w_t(\Lambda^{(m)}, \mathrm{Tr}) w_t(\Lambda^{(m)}, -\mathrm{Tr}) = d_t(\Lambda^{(m)})$ and so we obtain

$$w_t\big(\Lambda^{(0)}, \mathrm{Tr}\big) = w_t\big(\Lambda^{(m)}, \mathrm{Tr}\big) d_t\big(M_0 - M_1 - \Lambda^{(m)}\big), \tag{2}$$

which we will use for m even. Also

$$w_t\big(\Lambda^{(0)}, \mathrm{Tr}\big) = w_t\big(\Lambda^{(m)}, -\mathrm{Tr}\big)^{-1} d_t(M_0 - M_1), \tag{3}$$

which we will use for m odd. Furthermore $I^{(h)D} = \mathcal{G}^{(h)}$, so that $d_t(I^{(h)}) = d_t(\mathcal{G}^{(h)})$ and hence by the basic exact sequence (1)

$$d_t\big(\Lambda^{(h)}\big) d_t\big(\Lambda^{(h+1)}\big) = d_t\big(\mathcal{G}^{(h)}\big)^2. \tag{4}$$

To conclude we substitute (2) in (3) and (4).

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