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Travelling waves and dispersion relation in the spatial unfolding of a periodic orbit

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Abstract For a partial differential equation in spatial dimension one, admitting a spatially homogeneous time periodic solution, we show the generic existence, close to this solution, of a one-parameter family of travelling waves parametrized by their wave number k ($k = 0$ corresponding to the spatially homogeneous initial solution). The argument is elementary and relies on a direct application of singular perturbation theory (Fenichel's global center manifold theorem). ***To cite this article: E. Risler, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 833–838.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Ondes progressives et relation de dispersion dans le déploiement spatial d'une orbite périodique

Résumé Pour une équation aux dérivées partielles en dimension un d'espace, admettant une solution homogène en espace et périodique en temps, on montre l'existence, au voisinage de cette solution, d'une famille à un paramètre d'ondes progressives paramétrisées par leur nombre d'onde k ($k = 0$ correspondant à la solution spatialement homogène initiale). La justification, élémentaire, est basée sur un argument de perturbation singulière (théorème de la variété centrale globale de Fenichel). ***Pour citer cet article : E. Risler, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 833–838.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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On considère une équation aux dérivées partielles de la forme

$$u_t = F(u, \partial_x), \tag{1}$$

c'est-à-dire autonome (invariante par translation du temps) et homogène (invariante par translation d'espace). La variable champ u appartient à $\mathbf{R}^n$, $n \geqslant 2$, et la variable d'espace x appartient à $\mathbf{R}$ (dimension un d'espace). On suppose que cette équation admet une solution $u(x,t) = u_0(t)$ homogène en espace et périodique en temps (non constante) ; on note $T > 0$ sa plus petite période.

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L'objectif de cet article est de montrer que, sous des hypothèses très générales sur la forme de l'équation (1), et si des conditions génériques sont satisfaites, alors il existe une (unique) famille $(\omega_k, \Phi_k)_{k\simeq 0}$, dépendant continûment du paramètre k, avec $\omega_k \in \mathbf{R}$ et $\Phi_k : \mathbf{R} \to \mathbf{R}^n$ périodique de période 1, et vérifiant :

– $\omega_0 = 1/T$ et $\Phi_0(\omega_0 t) = u_0(t)$,
– pour tout $k \simeq 0$, l'onde progressive $(x, t) \mapsto \Phi_k(\omega_k t - kx)$ est une solution de l'équation (1).

En particulier, ceci définit la relation de dispersion (non linéaire) : $k \mapsto \omega_k$, $k \simeq 0$.

Posons $f(u) = F(u, 0)$; l'équation différentielle gouvernant les solutions spatialement homogènes de l'équation (1) s'écrit

$$u_t = f(u), \tag{2}$$

et on peut écrire :

$$F(u, \partial_x) = f(u) + C(u, \partial_x),$$

où le « terme de couplage » $C(u, \partial_x)$ vérifie $C(u, 0) = 0$ (il s'annule sur les fonctions homogènes). Notons $\alpha \in \mathbf{N}^*$ l'ordre de la dérivée spatiale la plus élevée qui intervient dans l'équation, et faisons les hypothèses suivantes.

(H1) L'entier α est pair.

(H′1) Le terme de couplage $C(u, \partial_x)$ est de la forme : $C(u, \partial_x) = C_1(u, \partial_x u, \ldots, \partial_x^{(\alpha-1)} u) + M(u)\partial_x^\alpha u$, où $C_1(., \ldots, .)$ vérifie $C_1(., 0, \ldots, 0) = 0$, et $M(u) \in \mathcal{M}_n(\mathbf{R})$ (toutes les dépendances sont régulières).

Notons $\mathcal{T}$ la trajectoire dans $\mathbf{R}^n$ de la solution $u_0(\cdot)$.

(H″1) Pour tout $u \in \mathcal{T}$, la matrice $M(u)$ n'a pas de valeur propre appartenant à $i\mathbf{R}$ (en particulier cette matrice est inversible).

Les hypothèses (H′1) et (H″1) garantissent l'existence d'un « système dynamique spatial » bien défini qui gouverne les solutions de type onde progressive, et les hypothèses (H1) and (H″1) vont garantir une condition d'« hyperbolicité normale » requise pour appliquer les arguments de perturbation singulière. Les hypothèses (H′1) et (H″1) sur la nature des termes de couplage sont probablement trop fortes, mais des hypothèses plus faibles semblent rendre plus problématique l'application des arguments de perturbation singulière.

On fait sur la solution $u_0(\cdot)$ l'hypothèse de transversalité suivante :

(H2) La solution périodique $t \mapsto u_0(t)$ a 1 comme multiplicateur de Floquet de multiplicité exactement un (c'est-à-dire qu'elle n'a pas de multiplicateur de Floquet égal à 1, à l'exception de celui canonique dans la direction du flot).

Notre résultat est le suivant.

THÉORÈME. – *Sous les hypothèses* (H1), (H′1), (H″1), *et* (H2), *il existe* $k_0 > 0$ *tel que, pour tout* $k \in]-k_0; k_0[$, *il existe un unique* ω_k *proche de* $\omega_0 = 1/T$, *et il existe une unique* (*à translation près*) *fonction* $\Phi_k : \mathbf{R} \to \mathbf{R}^n$, *périodique de période* 1, *proche de* $\Phi_0(\cdot) = u_0(\cdot\, T)$, *telle que l'application*

$$(x, t) \mapsto \Phi_k(\omega_k t - kx)$$

est une solution de l'équation (1).

La preuve, élémentaire, repose sur un argument de perturbation singulière : les ondes progressives apparaissent comme des solutions périodiques pour une perturbation singulière de l'équation différentielle (2). Un argument classique de perturbation singulière (variété centrale globale, théorème de Fenichel) permet de ramener cette perturbation singulière à une perturbation régulière, et le résultat en découle par le théorème de continuation de solutions périodiques de Poincaré.

We consider a partial differential equation of the form

$$u_t = F(u, \partial_x), \tag{3}$$

i.e., autonomous (invariant with respect to translations of time) and homogeneous (invariant with respect to translations of space). The field variable u belongs to $\mathbf{R}^n$, $n \geqslant 2$, and the space variable x belongs to $\mathbf{R}$ (spatial dimension one). We suppose that this equation admits a spatially homogeneous time periodic (nonconstant) solution $u(x, t) = u_0(t)$ (denote by $T > 0$ its smallest period).

The aim of this paper is to prove that, if some general assumptions on the form of Eq. (3) and some generic conditions are satisfied, then there exists a (unique) continuous family $(\omega_k, \Phi_k)_{k\simeq 0}$, $\omega_k \in \mathbf{R}$, $\Phi_k : \mathbf{R} \to \mathbf{R}^n$ periodic of period 1, satisfying:

- $\omega_0 = 1/T$ and $\Phi_0(\omega_0 t) = u_0(t)$,
- for any $k \simeq 0$, the travelling wave $(x, t) \mapsto \Phi_k(\omega_k t - kx)$ is a solution of Eq. (3).

In particular, this defines the (nonlinear) dispersion relation: $k \mapsto \omega_k$, $k \simeq 0$.

Let us write $f(u) = F(u, 0)$; the differential equation governing the spatially homogeneous solutions of Eq. (3) can be written

$$u_t = f(u), \tag{4}$$

and we can write:

$$F(u, \partial_x) = f(u) + C(u, \partial_x),$$

where the "coupling term" $C(u, \partial_x)$ satisfies $C(u, 0) = 0$ (it vanishes on homogeneous functions). Let us denote by $\alpha \in \mathbf{N}^*$ the order of the highest spatial derivative which is involved in the equation, and let us make the following hypotheses.

(H1) The integer α is even.

(H′1) The coupling term $C(u, \partial_x)$ takes the form: $C(u, \partial_x) = C_1(u, \partial_x u, \ldots, \partial_x^{(\alpha-1)} u) + M(u)\partial_x^{\alpha} u$, where $C_1(., \ldots, .)$ satisfies $C_1(., 0, \ldots, 0) = 0$, and $M(u) \in \mathcal{M}_n(\mathbf{R})$ (all dependences are smooth).

Let us denote by $\mathcal{T}$ the trajectory in $\mathbf{R}^n$ of the solution $u_0(\cdot)$.

(H″1) For any $u \in \mathcal{T}$, the matrix $M(u)$ has no eigenvalue in $i\mathbf{R}$ (in particular it is invertible).

Hypotheses (H′1) and (H″1) will ensure the existence of a properly defined "spatial dynamical system" governing travelling wave solutions, and hypotheses (H1) and (H″1) will guarantee a "normal hyperbolicity" condition required to apply the singular perturbation machinery. Hypotheses (H′1) and (H″1) on the structure of the coupling terms are probably too strong, but weaker hypotheses seem to require more involved arguments when applying singular perturbation arguments.

We make the following transversality assumption on the solution $u_0(\cdot)$:

(H2) The periodic solution $t \mapsto u_0(t)$ has 1 as a Floquet multiplier of multiplicity precisely one (i.e., it has no Floquet multiplier equal to 1 except the canonical one in the direction of the flow).

We can now state our result.

THEOREM. – *Under hypotheses* (H1), (H′1), (H″1), *and* (H2), *there exists* $k_0 > 0$ *such that, for any* $k \in]-k_0; k_0[$, *there exists a unique* ω_k *close to* $\omega_0 = 1/T$, *and there exists a unique* (*up to translation*) *map* $\Phi_k : \mathbf{R} \to \mathbf{R}^n$, *periodic of period* 1, *close to* $\Phi_0(\cdot) = u_0(\cdot\, T)$, *such that the map*

$$(x, t) \mapsto \Phi_k(\omega_k t - kx)$$

is a solution of Eq. (3).

The proof, which we now explain, relies on a singular perturbation argument. Travelling waves will appear as periodic solutions of a singular perturbation of Eq. (4). A classical singular perturbation argument (global central manifold, Fenichel's theorem), enables to reduce this singular perturbation to a regular perturbation. The result will follow by Poincaré continuation theorem for periodic solutions of ordinary differential equations.

Up to rescaling time, we can suppose that $T = 1$. We look for solutions of Eq. (3) of the form $u(x,t) = \phi(t - \varepsilon x)$, with ε close to 0 and $\phi(\cdot)$ close to the periodic solution $u_0(\cdot)$. Replacing into (3), we obtain (recall that α is even):

$$\varepsilon^\alpha \phi^{(\alpha)} = M(\phi)^{-1}\big(\phi' - f(\phi) - C_1\big(\phi, -\varepsilon\phi', \ldots, (-\varepsilon)^{\alpha-1}\phi^{(\alpha-1)}\big)\big) \tag{5}$$

(since the values of $\phi(\cdot)$ are supposed to be close to $\mathcal{T}$, according to hypothesis (H$''$1) the matrix $M(\phi)$ is invertible).

Eq. (5) is nothing else than a differential equation of order α in dimension n, which we wish to rewrite as a differential equation of order 1 in dimension $n\alpha$. Thus we shall introduce quantities $X, Y_1, \ldots, Y_{\alpha-1}$ (each being n-dimensional), corresponding to the derivatives of order $0, \ldots, \alpha - 1$ of ϕ. Besides, we wish to perform this rewriting in such a way that the quantity X appears as a "slow variable", and the quantity $Y = (Y_1, \ldots, Y_{\alpha-1})$ appears as a "fast variable". Thus we wish to choose the expressions of $(X, Y_1, \ldots, Y_{\alpha-1})$ in order to obtain an equation of the following form:

$$\dot{X} = P(X, Y, \delta), \qquad \delta\dot{Y} = Q(X, Y, \delta),$$

where δ is a small parameter (related to ε). This leads to the following definition for $(X, Y_1, \ldots, Y_{\alpha-1})$:

$$\begin{cases} X &= \phi, \\ Y_1 &= \phi', \\ Y_2 &= \delta\phi'', \\ \vdots & \vdots \\ Y_{\alpha-1} &= \delta^{\alpha-2}\phi^{(\alpha-1)} \end{cases}$$

and, according to (5), to the following expression for the differential equation governing these quantities:

$$\begin{cases} \dot{X} &= Y_1, \\ \delta\dot{Y}_1 &= Y_2, \\ \vdots & \vdots \\ \delta\dot{Y}_{\alpha-2} &= Y_{\alpha-1}, \\ \delta\dot{Y}_{\alpha-1} &= \dfrac{\delta^{\alpha-1}}{\varepsilon^\alpha} M(X)^{-1}\left(Y_1 - f(X) - C_1\left(X, -\varepsilon Y_1, \dfrac{\varepsilon^2}{\delta}Y_2, \ldots, \dfrac{(-\varepsilon)^{\alpha-1}}{\delta^{\alpha-2}}Y_{\alpha-1}\right)\right). \end{cases} \tag{6}$$

Thus we are led to choose $\delta = \varepsilon^{\alpha/(\alpha-1)}$, and we see that, in order to have a smooth dependence with respect to the small parameter in the expression of $\delta\dot{Y}_{\alpha-1}$, we have to introduce a new parameter, namely:

$$\gamma = \varepsilon^{1/(\alpha-1)} = \operatorname{sgn}(\varepsilon)|\varepsilon|^{1/(\alpha-1)}$$

(here $\operatorname{sgn}(\varepsilon)$ denotes the sign of ε, since α is even we have $\varepsilon = \gamma^{\alpha-1}$ and $\delta = \gamma^\alpha$). System (6) is written

$$\dot{X} = P(X, Y, \gamma), \qquad \gamma^\alpha\dot{Y} = Q(X, Y, \gamma), \tag{7}$$

with $P(X,Y,\gamma)=Y_1$ and

$$Q(X,Y,\gamma)=\big(Y_2,\dots,Y_{\alpha-1},M(X)^{-1}\big(Y_1-f(X)-C_1\big(X,-\gamma^{\alpha-1}Y_1,\gamma^{\alpha-2}Y_2,\dots,(-1)^j\gamma^{\alpha-j}Y_j,\\ \dots,(-1)^{\alpha-1}\gamma Y_{\alpha-1}\big)\big)\big).$$

We are now in position to apply a singular perturbation argument. The matrix $M(X)$ is invertible for X close enough to $\mathcal{T}$, say for X in a neighborhood Ω of $\mathcal{T}$, and P and Q depend smoothly on their arguments for $X\in\Omega$. When $\gamma=0$, the system (7) degenerates to the reduced "slow" system

$$\dot X=P(X,Y,0),\qquad 0=Q(X,Y,0).$$

For $X\in\Omega$, the equation $Q(X,Y,0)=0$ is equivalent to $(X,Y)\in\Sigma_0$, where $\Sigma_0\in\mathbf{R}^{n\alpha}$ is the graph of the map $H_0:\mathbf{R}^n\to\mathbf{R}^{n(\alpha-1)}$, $X\mapsto(f(X),0,\dots,0)$.

Rescaling time by a factor γ^α, system (7) becomes

$$X'=\gamma^\alpha P(X,Y,\gamma),\qquad Y'=Q(X,Y,\gamma),$$

which converges when $\gamma\to 0$ towards the "fast" system

$$X'=0,\qquad Y'=Q(X,Y,0),$$

for which the "slow" manifold Σ_0 consists entirely of equilibrium points.

For any $(X,Y)\in\Sigma_0$, we have

$$\frac{\partial Q}{\partial Y}(X,Y,0)=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ M(X)^{-1} & 0 & 0 & \dots & 0\end{pmatrix}. \tag{8}$$

The eigenvalues of this matrix are the (complex) roots of order $\alpha-1$ of the eigenvalues of the matrix $M(X)^{-1}$. Up to replacing Ω by a smaller neighborhood of $\mathcal{T}$, we can suppose, according to hypothesis (H″1), that, for $X\in\Omega$, no eigenvalue of $M(X)^{-1}$ belongs to $i\mathbf{R}$. As a consequence, since α is even (hypothesis (H1)), no eigenvalue of the matrix (8) belongs to $i\mathbf{R}$.

This provides the required hyperbolicity condition to apply Fenichel's global center manifold theorem [2,3]. According to this theorem, there exists a neighborhood I of 0 in $\mathbf{R}$, a neighborhood (still denoted by Ω) of $\mathcal{T}$ in Ω, and a smooth map $H:I\times\Omega\to\mathbf{R}^{n(\alpha-1)}$ such that, if for $\gamma\in I$ we denote by Σ_γ the graph of the map $H_\gamma:X\mapsto H(\gamma,X)$, then the following holds: for $\gamma\in I\setminus\{0\}\Sigma_\gamma$ is locally invariant under the dynamics of (7), and, for $\gamma=0$, Σ_0 and H_0 coincide with the notations above. As in local center manifold theory, "locally invariant" means that any solution of (7) which starts on Σ_γ remains on Σ_γ as long as X remains in Ω. The manifold Σ_γ is not unique, but it must contain the trajectory of any solution of (7) for which $X\in\Omega$ for all time.

For $\gamma\in I$, let us denote by h_γ the map $\pi\circ H_\gamma$, where $\pi:\mathbf{R}^{n(\alpha-1)}\to\mathbf{R}^n$ denotes the projection over the first n components. Then we have $h_0(X)=f(X)$, the map h_γ depends smoothly on γ, and the restriction to Σ_γ of the system (7) is conjugated by H_γ to the equation

$$\dot X=h_\gamma(X). \tag{9}$$

Let us write $\phi_0(t)=u_0(t)$, $t\in\mathbf{R}$. This function is a periodic solution of Eq. (9) for $\gamma=0$. According to hypothesis (H2) (transversality of this periodic solution), by Poincaré continuation theorem, for any γ

close enough to 0 Eq. (9) admits a unique solution $t \mapsto \phi_\gamma(t)$ which is close to $t \mapsto \phi_0(t)$ (uniqueness is up to time translation, and "close" means that trajectories and periods are close). As mentioned above, this periodic solution does not depend on the choice of the global center manifold Σ_γ, therefore it depends only on γ (for any $\gamma \simeq 0$, it is unique). Moreover we can choose the time parametrization so that $\phi_\gamma(t)$ depends smoothly on γ.

Let us mention that this result of continuation of a periodic solution under a singular perturbation was known before Fenichel, see, for instance, Anosov's paper [1]. However, Anosov's results do not state the regularity with respect to the parameter, which is required below.

Let us denote by $T(\gamma)$ the period of $t \mapsto \phi_\gamma(t)$, and let us write

$$\tilde{\phi}_\gamma(\xi) = \phi_\gamma\big(T(\gamma)\xi\big), \quad \xi \in \mathbf{R}.$$

The function $\tilde{\phi}_\gamma(\cdot)$ has period 1, and the travelling wave:

$$(x,t) \mapsto \phi_\gamma\big(t - \gamma^{\alpha-1}x\big) = \tilde{\phi}_\gamma\left(\frac{t}{T(\gamma)} - \frac{\gamma^{\alpha-1}}{T(\gamma)}x\right)$$

is a solution of (3).

Write $k(\gamma) = \gamma^{\alpha-1}/T(\gamma)$. For γ close to 0, we have (using the smoothness of $T(\gamma)$ with respect to γ):

$$\frac{\mathrm{d}k}{\mathrm{d}\gamma}(\gamma) = (\alpha-1)\gamma^{\alpha-2}\big(1 + \mathrm{o}(1)\big),$$

which shows that the map $\gamma \mapsto k(\gamma)$ is strictly increasing (and thus invertible) on a neighborhood of 0. Let us denote by $k \mapsto \gamma(k)$ the inverse, and (for k close enough to 0) let us write

$$\omega_k = \frac{1}{T(\gamma(k))} \quad \text{and} \quad \Phi_k(\xi) = \tilde{\phi}_{\gamma(k)}(\xi), \quad \xi \in \mathbf{R}.$$

Then the previous travelling wave is given by

$$(x,t) \mapsto \Phi_k(\omega_k t - kx),$$

and the theorem is proved.

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References

[1] D.V. Anosov, On limit cycles in systems of differential equations with a small parameter in the highest derivatives, Amer. Math. Soc. Transl. Sér. 2 33 (1963) 233–275; Translation from Mat. Sb. (N.S.) 50 (92) (1960) 299–334.

[2] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, J. Differential Equations 31 (1979) 53–98.

[3] C. Jones, Geometric singular perturbation theory, in: Lecture Notes in Math., Vol. 1609, 1995, pp. 44–118.