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Randomizing properties of convex high-dimensional bodies and some geometric inequalities

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Abstract Properties of convex bodies related to uniform distribution are studied. In particular, a low bound for the norm of the sum of independent geometrically distributed vectors is obtained. It extends the previously studied case of identically distributed vectors by Bourgain, Meyer, Milman and Pajor and solves a problem raised there. Another corollary asserts that any finite dimensional normed space has a "random cotype 2". ***To cite this article: E. Gluskin, V. Milman, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 875–879.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Les propriétés aléatoires des corps convexes de grande dimension et une inégalité géométrique

Résumé Nous étudions les propriétés des corps convexes liées à la distribution uniforme. En particulier, nous démontrons une borne inférieure pour la norme d'une somme de vecteurs aléatoires distribués géometriquement. Cette borne généralise le cas des vecteurs ayant la même distribution déjà étudié par Bourgain, Meyer, Milman et Pajor, et résout un problème posé par eux. Un autre corollaire énonce que chaque espace normé de dimension finie est de « cotype 2 aléatoire ». ***Pour citer cet article : E. Gluskin, V. Milman, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 875–879.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit K un corps convexe symétrique par rapport à l'origine dans $\mathbb{R}^n$, qui est assez loin d'un ellipsoide. Est-il possible de discerner cette différence par une caractéristique aléatoire de K ? Il est conjecturé qu'une séparation polynomiale en la dimension n'est pas possible. Nous étudions ici, de ce point de vue, quelques caractéristiques liées au concept de cotype. Il est observé qu'au moins dans ce cas là, la conjecture est vérifiée.

Il y a quelques structures aléatoires assez naturelles liées à un plongement du corps K dans $\mathbb{R}^n$. Dans ce cadre nous étudions les propriétés métriques (par rapport à K) d'une rotation aléatoire uA d'un ensemble fini $A \subseteq S^{n-1}$. Une autre structure aléatoire revient à la distribution uniforme (volume) sur K. Nous la

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désignerons sous le terme de « probabilité géometrique ». Quelques propriétés des séries finies de vecteurs aléatoires indépendants distribués géometriquement sont étudiées à la Section 2.

Nous utilisons les notations suivantes. Pour un ensemble mesurable $T \subseteq \mathbb{R}^n$ nous désignons par $|T|$ le volume de T. D est la boule Euclidienne standard, et $S^{n-1} = \partial D$ est la sphère Euclidienne. Etant donné un corps convexe K, symétrique par rapport à l'origine, nous notons $\|\cdot\|_K$ la norme dont la boule unité est K et $X_K = (\mathbb{R}^n, \|\cdot\|_K)$.

Le premier résultat que nous voudrions mettre en avant est le théorème suivant, qui réunit les Théorèmes 1 et 2 de la version Anglaise. La démonstration de ce théorème revient aux inégalités du type Brascamp–Lieb.

THÉORÈME. – *Soient $T_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, m$, des ensembles mesurables tels que $|T_i| = |D|$ pour tout i, et K un corps convexe symétrique par rapport à l'origine vérifiant la même condition $|K| = |D|$. Alors pour tous scalaires λ_i et tout t positif, on a l'inégalité suivante*

$$\mathrm{Prob}\left\{ (x_i)_{i=1}^m \in \prod_{i=1}^m T_i \;\middle|\; \left\| \sum_{i=1}^m \lambda_i x_i \right\|_K < t \right\} \leqslant \mathrm{Prob}\left\{ (x_i)_{i=1}^m \in D^m \;\middle|\; \left\| \sum_{i=1}^m \lambda_i x_i \right\|_D < t \right\} \leqslant \left(t \sqrt{\frac{\mathrm{e}}{\Sigma \lambda_i^2}} \right)^n.$$

Quelques cas particuliers de cette inégalité sont spécialement intéressants. Par example, si $T_1 = T_2 = K$ et $\lambda_1 = -\lambda_2 = 1$, $\lambda_i = 0$, $i \geqslant 3$, la première inégalité doit être comparée à [1]. Nous le ferons dans la Section 2.

Observons une conséquence géometrique importante.

COROLLAIRE 1. – *Soient $T_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, m$, et K un corps convexe symétrique par rapport à l'origine tels que $|T_i| = |K| = |D|$. Considérons la norme symétrique inconditionnelle suivante sur $\lambda = (\lambda_i \in \mathbb{R})_{i=1}^m$*

$$|\|\lambda\|| = \int_{T_1} \cdots \int_{T_m} \left\| \sum_1^m \lambda_i x_i \right\|_K \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_m}{|D|^m}. \tag{1}$$

Il existe une constante universelle $c > 0$ indépendante de n, T_i, et K telle que

$$|\|\lambda\|| \geqslant c \sqrt{\sum_1^m \lambda_i^2}. \tag{2}$$

L'intégrale (1) était étudiée dans [6], mais la borne (2) n'y était que conjecturée, et seulement une borne plus faible y était démontrée.

Un autre corollaire du théorème dit essentiellement que tout espace normé de dimension finie a un cotype 2 « aléatoire ». Précisément :

COROLLAIRE 2. – *Il existe une constante universelle $c > 0$ telle que pour tout n entier positif et tout espace normé $X = (\mathbb{R}^n, \|\cdot\|_K)$, les variables aléatoires uniformes $\{x_i \in K\}_{i=1}^n$ vérifient l'inégalité suivante avec une probabilité exponentiellement proche de 1 pour chaque $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$.*

$$\operatorname*{Ave}_{\pm} \left\| \sum_1^n \pm \lambda_i x_i \right\|_K \geqslant c \sqrt{\sum_1^n |\lambda_i|^2}.$$

Dans le cas où la structure Euclidienne est fixée dans $\mathbb{R}^n$, nous démontrons à la Section 3, que, sous certaines conditions de non-degénérescence, tout espace normé a — en un sens précis (*voir* Theorem 3) — en même temps un type et un cotype aléatoires par rapport à la structure induite par la structure Euclidienne.

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1. Introduction and notation

Let K be a convex centrally symmetric body in $\mathbb{R}^n$, which is rather far from an ellipsoid. Is it possible to detect this difference through some random ideal characteristic of K? A conjecture by the second author is that for a power scale (with respect to dimension), the answer is no. We study here, from this point of view, some of the characteristics related to the notion of cotype. It is observed that at least in these cases the conjecture is true. There are a few fairly natural random structures related to an embedding of a body K in $\mathbb{R}^n$. One of them is induced by the Euclidean structure of $\mathbb{R}^n$. Under this setting for a given finite subset $A \subset S^{n-1}$ the metric (with respect to K) properties of a random rotation uA of the set A are investigated. Another random structure is induced by a uniform (volume) distribution on K and will be referred to as geometric probability. Some properties of finite series of independent random vectors with geometric distribution are investigated in Section 2.

We will use the following notation. For a measurable set $T \subset \mathbb{R}^n$ we write $|T|$ for a Lebesgue volume of T, $\mathcal{D}$ is the standard Euclidean ball, $S^{n-1} = \partial\mathcal{D}$ is the Euclidean sphere and $|x|$ is the standard Euclidean norm of a vector $x \in \mathbb{R}^n$. For a centrally symmetric convex body K we define $\|\cdot\|_K$ as the norm with the unit ball K and $X_K = (\mathbb{R}^n, \|\cdot\|_K)$. We also denote $M(X) = \sqrt{\int_{S^{n-1}} \|x\|_K^2 \,\mathrm{d}\sigma(x)}$ where $\sigma(x)$ is the probability rotation invariant measure.

2. Uniform volume distribution

Let $K \subset \mathbb{R}^n$ and $T_i \subset \mathbb{R}^n$, $i = 1, \ldots, m$, be some given bodies. We are going to estimate the probability of the event $\|\sum \lambda_i x_i\|_K < t$, where the vectors x_i run independently over T_i which is equipped with a geometric probability, and λ_i and t are some given numbers. Firstly, we observe that the case of Euclidean balls is an extremal one for this problem. Namely, as a consequence of the Brascamp–Lieb–Luttinger inequality [5] the following fact is true.

THEOREM 1. – *Let $T_i \subset \mathbb{R}^n$, $i = 1, \ldots, m$, be some measurable sets s.t. $|T_i| = |D|$ for any i, and K be a convex symmetric body satisfying the same condition $|K| = |D|$. Then for any scalars λ_i and for any positive t the following inequality holds*

$$\mathrm{Prob}\left\{ (x_i)_{i=1}^m \in \prod_{i=1}^m T_i \;\middle|\; \left\|\sum_{i=1}^m \lambda_i x_i\right\|_K < t \right\} \leqslant \mathrm{Prob}\left\{ (x_i)_{i=1}^m \in D^m \;\middle|\; \left\|\sum_{i=1}^m \lambda_i x_i\right\|_D < t \right\}.$$

The last quantity can be written as an iterated integral of the Bessel functions. It can probably be estimated directly although we do not see a clear way of doing so. Fortunately, the general theory provides us with pretty good bounds. Firstly, from the concentration phenomenon on the products $S^{n-1} \times \cdots \times S^{n-1}$ ([7], *see* [8]) it follows that some constants c and c' exist s.t. for t satisfying $1/2 \leqslant t \leqslant 1 - c'/\sqrt{n}$, the probability in the theorem is bounded by t^{cn}.

A better estimate is obtained using the Brascamp–Lieb inequality.

THEOREM 2. – *Let K be some convex symmetric body and T_i, $i = 1, \ldots, m$, be some measurable sets s.t. $|T_i| = |K|$, for any i. Then for any positive t and reals λ_i, the following inequality holds*

$$\mathrm{Prob}\left\{ (x_i)_{i=1}^m \in \prod_{i=1}^m T_i : \left\|\sum_{i=1}^m \lambda_i x_i\right\|_K < t\sqrt{\sum_{i=1}^m \lambda_i^2} \right\} \leqslant (t\sqrt{\mathrm{e}})^n.$$

Applying the Brascamp–Lieb inequality to such problems goes back to Ball [2]. We extend his method to the multivariable case by using tensorisation of the Ball inequality given by F. Barthe [3].

The following is an immediate consequence of Theorem 2

$$\mathrm{Prob}_{(x_i)\in\Pi T_i}\left\{\min_{\varepsilon_i=\pm1}\left\|\sum_1^m \varepsilon_i\lambda_i x_i\right\|_K \geqslant t\sqrt{\sum_1^m \lambda_i^2}\right\} \geqslant 1-(t\sqrt{\mathrm{e}})^n\cdot 2^m.$$

The bound given by Theorem 2 is very good for small t but certainly does not work for $1>t>\mathrm{e}^{-1/2}$. For such t just the concentration phenomenon gives a better estimate, especially for small k. For example, for $k=2$, one has

$$\mathrm{Prob}\left\{(x_1,x_2)\in T_1\times T_2: \|\lambda_1x_1+\lambda_2x_2\|_K < t\sqrt{\lambda_1^2+\lambda_2^2}\right\} \leqslant 3\,\mathrm{e}^{-(1-t)^2n/2}.$$

For $\lambda_1=-\lambda_2=1/\sqrt{2}$, $T_1=T_2=K$ the last inequality is a slightly weaker version of the Arias-de-Reyna–Ball–Villa result [1]. In fact the [1] approach can be used for any number of bodies. Namely, the following is true.

PROPOSITION. – *For any K and T_i, $i=1,\dots,m$, as above the following inequality holds*

$$\mathrm{Prob}\left\{(x_i)_{i=1}^m\in\prod_{i=1}^m T_i : \left\|\sum_{i=1}^m x_i\right\|_K < t\sqrt{m}\right\} \leqslant t^n\left[1+\frac{1-t^2}{m-1}\right]^{n(m-1)/2} \leqslant t^n \exp\frac{n}{2}\left(1-t^2\right).$$

Let us point out that the last expression is bounded by $\mathrm{e}^{-n(1-t)^2}$.

Also a Corollary of Theorem 1 is the following geometric inequality:

COROLLARY 1. – *Let $T_i\subset\mathbb{R}^n$, $i=1,\dots,m$, and K be convex centrally symmetric bodies and $|T_i|=|K|=|\mathcal{D}|$. Consider the following unconditional symmetric norm on $\lambda=(\lambda_i\in\mathbb{R})_{i=1}^m$*

$$|||\lambda||| = \int_{T_1}\cdots\int_{T_m}\left\|\sum_1^m\lambda_i x_i\right\|_K \frac{\mathrm{d}x_1\cdots\mathrm{d}x_m}{|\mathcal{D}|^m}. \qquad (*)$$

Then there is a universal constant $c>0$, independent of n, T_i and K in $\mathbb{R}^n$, such that

$$|||\lambda||| \geqslant c\sqrt{\sum_1^m\lambda_i^2}. \qquad (**)$$

The integral $(*)$ was studied in [6] but the bound $(**)$ was only conjectured there and a weaker bound proved.

Another corollary of Theorem 1 essentially states that any finite dimensional normed space has a "random" cotype 2:

COROLLARY 2. – *There is a universal constant $c>0$ such that for every integer n and any normed space $X=(\mathbb{R}^n,\|\cdot\|_K)$, random iid uniform in K variables $\{x_i\in K\}_{i=1}^n$ satisfy with exponentially close to 1 probability for any $\lambda_i\in\mathbb{R}$, $i=1,\dots,n$,*

$$\mathop{\mathrm{Ave}}_{\pm}\left\|\sum_1^n \pm\lambda_i x_i\right\|_K \geqslant c\sqrt{\sum_1^n|\lambda_i|^2}.$$

Moreover for $m \geqslant n$ with exponentially $($in $n)$ close to 1 geometric probability on K^m, one has

$$\sigma_m\left\{(\lambda_i) \in S^{m-1} : \left\|\sum_1^m \lambda_i x_i\right\|_K < t\right\} \leqslant (ct)^n,$$

where c is some universal constant.

3. Spherical uniform distribution

We will consider now similar randomized properties but with respect to randomness associated with a fixed Euclidean structure.

Let $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$ be a normed space also equipped with a Euclidean norm $|\cdot|$. We need a condition of "non-degeneration" between two norms, the norm $\|\cdot\|$ of X and the Euclidean norm $|\cdot|$. Note that

$$b(X) := \max\{\|x\| \mid |x| = 1\} \Big/ \sqrt{\int_{S^{n-1}} \|x\|^2 \, \mathrm{d}\sigma(x)} \leqslant \sqrt{n}$$

and $b(X)$ being close to $\sqrt{n}$ means degeneration. For a given $\|\cdot\|$ one may always find a Euclidean norm $|\cdot|$ such that for some $c > 0$

$$b(X) \leqslant c\sqrt{n/\log n} \qquad (***)$$

and this is a condition of non-degeneration of $\|\cdot\|$ with respect to the Euclidean norm $|\cdot|$.

THEOREM 3. – *There is a universal constant $c_0 > 0$ such that, for any n and $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$ which satisfies the non-degeneration condition $(***)$ with a constant c_0, the following is true*:

Consider any set $\{x_i\}_1^n \subset S^{n-1}$; there is an orthogonal operator $u \in \mathrm{O}(n)$ such that for any $\lambda_i \in \mathbb{R}^n$

$$\frac{1}{2}\sqrt{\sum_1^n \lambda_i^2} \cdot M(X) \leqslant \operatorname*{Ave}_{\varepsilon_i = \pm 1; \pi \in \Pi_n} \left\|\sum_1^n \varepsilon_i \lambda_{\pi(i)} u x_i\right\| \leqslant 2\sqrt{\sum_1^n \lambda_i^2} M(X).$$

Here Π_n is the permutation group and the result is true for a large measure of orthogonal operators "u".

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