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Regularity results for electrorheological fluids: the stationary case

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Abstract We report on some regularity results for weak solutions to systems modelling electrorheological fluids in the stationary case, as proposed in [8]. ***To cite this article: E. Acerbi, G. Mingione, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 817–822.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Résultats de régularité pour les fluides électrorhéologiques : le cas stationnaire

Résumé On prouve des résultats de régularité pour les solutions faibles de systèmes modélisant les fluides électrorhéologiques dans le cas stationnaire, utilisant le modèle introduit dans [8]. ***Pour citer cet article : E. Acerbi, G. Mingione, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 817–822.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Les fluides électrorhéologiques ont étés récemment l'objet de nombreuses études : ceux-ci sont en effet très spéciaux car leurs propriétés mécaniques changent en fonction du champ électromagnétique **E**, parfois de façon « dramatique ». De nombreux auteurs ont abordé la modélisation mathématique de tels fluides sous différents points de vue, ceci conduisant à diverses approches mathématiques et numériques (*voir* l'introduction de [8] et les références qui y sont mentionnées).

Dans le contexte de la mécanique continue, ces fluides ont la propriété d'être non Newtoniens ; très récemment, Rajagopal et Růžička (*voir* [8,7]) ont développé un très intéressant modèle mathématique pour de tels fluides, celui ci prenant en compte la délicate interaction entre le champ électromagnétique **E** et le liquide en mouvement [8,9]. Une caractéristique du système correspondant (*voir* [9]) est que le champ de vecteurs S doit vérifier (par hypothèse) des conditions de croissance non standard via l'exposant $p \equiv p(|\mathbf{E}|^2)$, voir (6) et (10). Ceci traduit la dépendance entre les propriétés mécaniques du fluide et le champ électromagnétique. Nous nous intéressons ici au système stationnaire (8) pour les fluides électrorhéologiques ainsi qu'à la régularité des solutions faibles. On suppose (*voir* [8,9]) que l'exposant

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$p:\Omega\to(1,+\infty)$ dépend juste de la variable x. Le système (1) étant découplé, ceci ne nuit pas à la généralité de ce travail.

Notre résultat principal est :

THÉORÈME 1. – *Soit* $u\in W^{1,p(x)}_{\mathrm{loc}}$ *une solution faible du système*

$$-\operatorname{div} S\big(x,\mathbf{E},\varepsilon(u)\big)+D\pi+[Du]u=f+\chi^{\mathbf{E}}[D\mathbf{E}]\mathbf{E},\quad \operatorname{div} u=0. \tag{1}$$

Si de plus on suppose que l'exposant $p:\Omega\to\mathbf{R}^+$ *est hölderien et tel que* :

$$\frac{9}{5}<\gamma_1\leqslant p(x)\leqslant\gamma_2<+\infty, \tag{2}$$

alors il existe un ouvert $\Omega_0\subset\Omega$ *tel que* Du *est hölderien sur* Ω_0 *et* $|\Omega\setminus\Omega_0|=0$.

Il est important de noter que la borne inférieure apparaissant dans (2) est la même que celle obtenue par Růžička pour prouver l'existence de solutions faibles, alors qu'il n'y a pas de borne supérieure pour γ_2, ce qui permet de traiter une grande classe de fluides.

La prochaine étape sera d'utiliser ce résultat pour trouver des estimations de la dimension de Hausdorff de l'ensemble singulier $\Omega\setminus\Omega_0$.

La démonstration utilise une procédure de blow-up ainsi qu'une localisation convenable afin de surmonter les difficultés dues à la condition de croissance non standard (*voir* [5,6]). On utilise, en particulier, des inégalités de Korn et des inégalités de type Hölder inverse dans les espaces de Orlicz.

De plus, les techniques développées ici s'étendent en dimension n quelconque à des systèmes plus généraux de type :

$$-\operatorname{div} A(x,Du)=B(x,u,Du)$$

et plus généralement,

$$-\operatorname{div} A\big(x,\varepsilon(u)\big)=B(x,u,Du),\quad \operatorname{div} u=0, \tag{3}$$

satisfaisant des conditions de croissance non standard. Pour ce type de système, les résultats de régularité obtenus sont nouveaux. Les démonstrations complètes vont paraître en [2].

1. Introduction

In recent years some attention has been paid to the study of electrorheological fluids; these are very special fluids possessing the ability to change even in a dramatic way their mechanical properties when in presence of an electromagnetic field **E** (their viscosity may vary by a factor of 1000). The mathematical modelling of such fluids was investigated by different authors adopting different points of view and involving various mathematical and numerical approaches (*see* the introduction of [8] and the references therein). In the context of continuum mechanics these fluids are seen as non-Newtonian fluids; very recently Růžička (following the ideas proposed by Rajagopal and Růžička in [7]) developed an interesting mathematical model for such fluids, keeping into account the delicate interaction between the electromagnetic field **E** and the moving liquid. The resulting system (*see* [8] for a description of the building procedures and for a general analysis) coming up from these studies is:

$$\begin{aligned}
&\operatorname{curl}\mathbf{E}=0,\qquad \operatorname{div}\mathbf{E}=0,\\
&u_t-\operatorname{div} S\big(\mathbf{E},\varepsilon(u)\big)+D\pi+[Du]u=f+\chi^{\mathbf{E}}[D\mathbf{E}]\mathbf{E},\\
&\operatorname{div} u=0,
\end{aligned} \tag{4}$$

where, according to the notations proposed in [8,9], $u:\Omega(\subset \mathbf{R}^3)\to\mathbf{R}^3$ is the velocity, $\mathbf{E}$ is the applied electromagnetic field, S the extra stress tensor, π the pressure and $\chi^{\mathbf{E}}$ the constant dielectric suspectibility, and following a standard notation, $\varepsilon(u)$ denotes the symmetric part of the gradient.

The constitutive relation proposed in [7–9] for the extra stress S is the following:

$$S(\mathbf{E},z):=\alpha_{21}\big[\big(1+|z|^2\big)^{(p-1)/2}-1\big]\mathbf{E}\otimes\mathbf{E}+\big(\alpha_{31}+\alpha_{33}|\mathbf{E}|^2\big)\big(1+|z|^2\big)^{(p-2)/2}z$$
$$+\alpha_{51}\big(1+|z|^2\big)^{(p-2)/2}(z\mathbf{E}\otimes\mathbf{E}+\mathbf{E}\otimes z\mathbf{E}) \tag{5}$$

for any $z\in\mathcal{S}_n$, the space of symmetric $n\times n$ matrices, with $n=3$.

The main new feature of system (4) is that the monotonicity and the ellipticity properties of the vector field S are strongly influenced by $\mathbf{E}$ through a variable growth exponent dependence: indeed in (5) the exponent p is actually a function of the quantity $|\mathbf{E}|^2$. Since (4) is uncoupled, one may first obtain $\mathbf{E}=\mathbf{E}(x)$, thus the dependence of p and S on $\mathbf{E}$ is indeed a dependence on x. With a suitable choice of the parameters $\alpha_{21},\alpha_{31},\alpha_{33},\alpha_{51}$ it turns out that:

$$D_zS(x,z)\lambda\otimes\lambda\geqslant\nu\big(1+\big|\mathbf{E}(x)\big|^2\big)\big(1+|z|^2\big)^{(p(x)-2)/2}|\lambda|^2, \tag{6}$$

$$\big|D_zS(x,z)\big|\leqslant L\big(1+|z|^2\big)^{(p(x)-2)/2}, \tag{7}$$

for any symmetric 3×3 matrices z, λ, where the function $p:\mathbf{R}^+\to(1,+\infty)$ reflects the physical properties of the fluid and has in general large oscillations when $|\mathbf{E}|$ changes (with $p<2$ when $|\mathbf{E}|$ is large).

We are interested in the regularity of solutions to systems similar to (4) in the stationary case, when it reduces to

$$-\operatorname{div}S\big(x,\varepsilon(u)\big)+D\pi+[Du]u=g(x),\quad \operatorname{div}u=0, \tag{8}$$

where $g\equiv f+[D\mathbf{E}(x)]\mathbf{E}(x)$.

The analysis and the existence theory for the previous system has been established in [8], and we concentrate on the regularity of weak solutions.

The natural energy associated with this problem is given by

$$\int_\Omega\big|\varepsilon(u)\big|^{p(x)}\,\mathrm{d}x,$$

or by its "full" version

$$\int_\Omega|Du|^{p(x)}\,\mathrm{d}x, \tag{9}$$

and this fact makes it natural to introduce the following generalization of the usual Sobolev space:

$$\mathrm{W}^{1,p(x)}:=\big\{u\in\mathrm{W}^{1,1}\big(\Omega;\mathbf{R}^3\big):\big|\varepsilon(u)\big|^{p(x)}\in\mathrm{L}^1(\Omega)\big\}.$$

At this stage we define the weak solutions (*see* [8], Chapter 3):

DEFINITION. – Let $\Omega\subset\mathbf{R}^3$ be a bounded domain. A function $u\in\mathrm{W}^{1,p(x)}_{\mathrm{loc}}$ is a weak solution to system (8) iff $\operatorname{div}u=0$ and for any smooth vector field φ with $\operatorname{div}\varphi=0$

$$\int_\Omega A\big(x,\varepsilon(u)\big)\varepsilon(\varphi)\,\mathrm{d}x=\int_\Omega B(x,u,Du)\varphi\,\mathrm{d}x.$$

For later convenience we have set $A=S$ and $B=-[Du]u+g(x)$.

As is clear from (6), a major difficulty to be overcome is the fact that S exhibits a nonstandard growth (*see* [1] and the references therein), that is its growth and coercivity exponents are different:

$$L^{-1}\big(|z|^{\gamma_1}-1\big)\leqslant S(x,z)z\leqslant L\big(|z|^{\gamma_2}+1\big), \tag{10}$$

where $\gamma_1 := \min p(x) < \gamma_2 := \max p(x)$.

Our main result is concerned with partial regularity of solutions to system (8).

THEOREM 2. – *Let $u \in \mathrm{W}^{1,p(x)}_{\mathrm{loc}}$ be a weak solution to system* (8) *and suppose that the exponent $p:\Omega\to\mathbf{R}^+$ is a Hölder continuous function such that*

$$\frac{9}{5}<\gamma_1\leqslant p(x)\leqslant\gamma_2<+\infty. \tag{11}$$

Then there exists an open set $\Omega_0\subset\Omega$ such that $|\Omega\setminus\Omega_0|=0$ and Du is Hölder continuous in Ω_0.

Up to our knowledge this is, beside the higher differentiability result obtained in [8] (*see* also [10] for periodic boundary conditions), the first regularity result for the model of electrorheological fluids proposed in [8], and in any case the first in a pointwise sense. A further step, based on Theorem 2, will be the estimate of the Hausdorff dimension of the singular set $\Omega\setminus\Omega_0$.

We add some comments; first, note that of special importance in the theory are the bounds (11) allowed for $p(x)$: these reflect the physical properties of a fluid. It is customary to assume that $1<\gamma_1\leqslant p(x)\leqslant\gamma_2$; of course the larger is the interval $[\gamma_1,\gamma_2]$ the larger is the class of fluids the model is going to cover. In other words the amplitude $\gamma_2-\gamma_1$ describes the possible excursions of the viscosity of the fluid when $\mathbf{E}$ changes, so it is important to prove results allowing for large values of $\gamma_2-\gamma_1$.

In [8] the author proves existence of weak solutions for the stationary problem under the only hypothesis (11), γ_2 being an arbitrarily large number (let us note that the same lower bound also appears when treating non-Newtonian fluids of standard type, that is when p is a constant; for these issues see [4]). Subsequently different bounds (according to the type of problem under consideration) are introduced in [8] on γ_2 and sometimes also on γ_1 in order to prove existence of higher differentiable solutions, for which $\gamma_1\geqslant 9/5$ is not sufficient anymore, in this way further restricting the class of fluids under consideration.

On the other hand, the hypotheses we consider here are consistent with, and in some respect weaker than, the ones considered by Růžička: in particular the lower bound (11) on γ_1 is the same found by Růžička, while there is no upper bound for γ_2, which is needed in [8] to prove existence of strong solutions: this allows to treat a broad class of fluids for which higher excursions of the viscosity (and consequently of $p(x)$) are observed. In this way the existence theorem of Růžička, for which (11) suffices, has now a regularity counterpart.

2. A more general case and ideas from the proof

The techniques used to obtain Theorem 2 are suitable for obtaining results for a more general class of systems, including the one of electrorheological fluids we considered. This is of interest for several reasons; in the paper [11] (following previous work by Baranger and Mikelic) Zhikov proposed a model for a class of fluids that are influenced in a similar way by the temperature T, rather than by an external electromagnetic field $\mathbf{E}$. In this model, once again, the stress tensor satisfies growth conditions of the type (6), (7), that we may call of $p(x)$ type, and the underlying energy is (9). The exponent function $p(x)\equiv p(T(x))$ turns out to be also an unknown of the system (which is highly coupled), thus minimal regularity assumptions must be considered on it: both in Theorem 2 and in the following Theorem 3 we allow $p(x)$ to be simply Hölder continuous.

Our results allow to treat a large class of systems with nonstandard growth conditions, that were not covered by the available regularity theory (*see* [5,6]); this may be of interest in itself, also looking at the large number of papers that recently appeared on the issue (*see* [5,6] and the references therein).

We briefly describe our results in some detail; we deal with systems of the following kind:

$$-\operatorname{div} A\big(x, \varepsilon(u)\big) + D\pi = B(x, u, Du), \quad \operatorname{div} u = 0, \tag{12}$$

where this time $\Omega \subset \mathbf{R}^n$ and the continuous vector fields $A : \Omega \times \mathcal{S}_n \to \mathbf{R}^{n^2}$ and $B : \Omega \times \mathbf{R}^n \times \mathbf{R}^{n^2} \to \mathbf{R}^n$ satisfy the following growth and ellipticity assumptions:

$$A(x, \cdot) \in C^1(\mathcal{S}_n), \tag{H1}$$

$$\big|DA(x,z)\big| \leqslant L\big(1+|z|^2\big)^{(p(x)-2)/2}, \tag{H2}$$

$$DA(x,z)\lambda \otimes \lambda \geqslant L^{-1}\big(1+|z|^2\big)^{(p(x)-2)/2}|\lambda|^2, \tag{H3}$$

$$\big|A(x,z) - A(x_0,z)\big| \leqslant L|x-x_0|^\alpha\big[\big(1+|z|^2\big)^{(p(x)-1)/2} + \big(1+|z|^2\big)^{(p(x_0)-1)/2}\big]\big(\log\big(1+|z|\big)+1\big), \tag{H4}$$

$$\big|B(x,u,\tilde{z})\big| \leqslant L\big(|u||\tilde{z}| + f(x)\big) \tag{H5}$$

for any $z, \lambda \in \mathcal{S}_n$, $x, x_0 \in \Omega$, $u \in \mathbf{R}^n$, $\tilde{z} \in \mathbf{R}^{n^2}$, where $1 \leqslant L < +\infty$, and $f : \Omega \to \mathbf{R}^+$ and $p : \Omega \to]1, +\infty[$ are functions such that

$$f \in \mathrm{L}^{n+n\beta}_{\mathrm{loc}}, \quad p(x) \geqslant \gamma_1 \geqslant \frac{3n}{n+2} + \beta, \tag{H6}$$

$$\big|p(x) - p(x_0)\big| \leqslant L|x-x_0|^\alpha \tag{H7}$$

for some $\beta > 0$ and $0 < \alpha < 1$. The notion of weak solution remains as in the definition above and the result available for such weak solutions is the same of Theorem 2:

THEOREM 3. – *Let* $u \in \mathrm{W}^{1,p(x)}_{\mathrm{loc}}$ *be a weak solution to system* (12) *such that* (H1)–(H7) *are satisfied. There exists an open set* $\Omega_0 \subset \Omega$ *such that* $|\Omega \setminus \Omega_0| = 0$ *and Du is Hölder continuous in* Ω_0.

Of course the same result is valid for the simpler case of systems like (3); we point out that regularity results for the variational case, i.e., the case of minimizers of the functional in (9), have been obtained, for instance, in [1,3] (see also the related references).

Finally we say a few words about the proofs. The starting point is proving a higher integrability result stating that actually $|Du|^{p(x)} \in \mathrm{L}^{1+\delta_1}$, for some $\delta_1 > 0$, rather than just $|Du|^{p(x)} \in \mathrm{L}^1$. This gives the manoeuvre room needed to adopt a blow-up procedure, which is a common tool when proving partial regularity, but a main point here is that the nonstandard growth conditions of the system force us to blow-up solutions not in the whole Ω but in small open subsets depending on the solution itself, on the higher integrability exponent δ_1, and on the size of the oscillations of $p(x)$. At this stage various higher integrability results are important to overcome the lack of standard growth conditions of the system, and in particular a quantitative knowlegde of the stability of certain higher integrability exponents coming up from reverse Hölder inequalities will be crucial. Moreover, in order to treat the physically important case $p(x) \leqslant 2$ we need to prove certain forms of Korn inequalities for a one parameter family of Orlicz spaces, paying attention to the stability of the constants appearing, uniformly with respect to the parameter. The regularity of the solutions is then achieved via a quite delicate localization of the iteration arguments employed to get partial regularity. The complete proofs will appear in [2].

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