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Santaló's inequality on $\mathbb{C}^n$ by complex interpolation

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Abstract A new approach to Santaló's inequality on $\mathbb{C}^n$ is obtained by combining complex interpolation and Berndtsson's generalization of Prékopa's inequality. ***To cite this article: D. Cordero-Erausquin, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 767–772.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Inégalité de Santaló sur $\mathbb{C}^n$ par interpolation complexe

Résumé On donne une nouvelle approche de l'inégalité de Santaló en combinant l'interpolation complexe et la généralisation de l'inégalité de Prékopa obtenue par Berntdsson. ***Pour citer cet article : D. Cordero-Erausquin, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 767–772.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Etant donné K un corps convexe symétrique de $\mathbb{R}^n$, si K° désigne le polaire de K, l'inégalité de Santaló [6] dit que

$$\mathrm{vol}(K)\,\mathrm{vol}\left(K^\circ\right) \leqslant \mathrm{vol}(D_n)^2, \tag{1}$$

où vol désigne la mesure de Lebesgue et D_n la boule euclidienne usuelle de $\mathbb{R}^n$ (on pourra consulter [4] pour avoir une preuve moderne basée sur la symétrisation de Steiner). Il est connu que l'inégalité (1) est liée de façon naturelle à l'interpolation complexe. En effet, lorsqu'on interpole entre un espace et son dual, on obtient « au milieu » (c'est-à-dire en $1/2$) l'espace euclidien. Par conséquent, l'inégalité de Santaló sur $\mathbb{C}^n$ serait une conséquence de la log-concavité du volume des boules unités des espaces interpolés. Cependant, cette approche n'a pas été développée car la question de la log-concavité est restée ouverte. L'outil principal manquant était une version complexe, où la convexité est remplacée par la plurisousharmonicité, de l'inégalité de Prékopa [5]. Cette version a été récemment obtenue par Berndtsson [2]. Dans cette Note, nous montrons comment l'inégalité de Berndtsson permet de retrouver, entre autres, l'inégalité de Santaló sur $\mathbb{C}^n$. Mais nous insistons sur le fait que le lien avec l'interpolation complexe n'est pas de nous, et que nous sommes redevable à Bernard Maurey et à Alain Pajor d'avoir attiré notre attention sur l'article de Berndtsson et sur son possible lien avec (1).

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Nous travaillons avec des espaces normés complexes de dimension finie n, c'est-à-dire avec des espaces du type $X=(\mathbb{C}^n,\|\cdot\|_X)$. Une *boule* de $\mathbb{C}^n$ est une boule unité $B_X:=\{w\in\mathbb{C}^n;\ \|w\|_X\leqslant 1\}$ pour une certaine norme $\|\cdot\|_X$ sur $\mathbb{C}^n$. Une boule de $\mathbb{C}^n$ est donc un convexe compact d'intérieur non vide invariant par l'action $\mathbb{C}^n\times\mathbb{R}\ni(w,\theta)\to \mathrm{e}^{\mathrm{i}\theta}w$. Ainsi une boule de $\mathbb{C}^n$ est un corps convexe symétrique de $\mathbb{R}^{2n}$, alors que la réciproque est fausse. Etant donné deux espaces $X=(\mathbb{C}^n,\|\cdot\|_X)$ et $Y=(\mathbb{C}^n,\|\cdot\|_Y)$ on note $[X,Y]_z=(\mathbb{C}^n,\|\cdot\|_z)$ l'espace interpolé entre X et Y en $z\in C:=\{z\in\mathbb{C}\mid 0\leqslant\Re(z)\leqslant 1\}$. Rappelons que $[X,Y]_z=[X,Y]_\theta$ avec $\theta=\Re(z)$. La particularité de la dimension finie est que l'on a toujours que $[X,Y]_z^*=[X^*,Y^*]_z$ où $Z^*=(\mathbb{C}^n,\|\cdot\|_*)$ désigne le dual de $Z=(\mathbb{C}^n,\|\cdot\|)$ dans la dualité $\langle w^1,w^2\rangle:=\sum_{k=1}^n w_k^1w_k^2$ et $\|w\|_*:=\sup_{\|w'\|\leqslant 1}|\langle w,w'\rangle|$. Comme conséquence, nous avons que, si

$$\mathcal{F}:=\Big\{f:C\longrightarrow\mathbb{C}^n\ \mid f \text{ continue bornée sur } C,\ \text{holomorphe sur } \Re(z)\in(0,1) \text{ et telle que}$$
$$\lim_{t\to\pm\infty} f(\alpha+\mathrm{i}t)=0 \text{ pour } \alpha=0 \text{ ou } 1\Big\}$$

et

$$B_N\big(X^*,Y^*\big):=\Big\{f\in\mathcal{F}\ \Big|\ \sup_{t\in\mathbb{R}}\big\|f(\mathrm{i}t)\big\|_{X^*}\leqslant 1 \text{ et } \sup_{t\in\mathbb{R}}\big\|f(1+\mathrm{i}t)\big\|_{Y^*}\leqslant 1\Big\},$$

alors pour tout $z\in C$ et tout $w\in\mathbb{C}^n$,

$$\|w\|_z=\sup_{f\in B_N(X^*,Y^*)}\big|\big\langle f(z),w\big\rangle\big|.$$

Nous montrons alors (Théorème 2.2) que la fonction

$$\begin{aligned}\mathbb{C}^n\times C&\longrightarrow\mathbb{R},\\ (w,z)&\longrightarrow\|w\|_z\end{aligned}$$

est plurisousharmonique, c'est-à-dire, que sa restriction à n'importe quelle droite complexe est sous-harmonique. Nous pouvons alors utiliser le résultat de Berndtsson (Théorème 3.1). Nous en déduisons que si μ est une mesure invariante par l'action $(w,\theta)\to \mathrm{e}^{\mathrm{i}\theta}w$, ayant une densité $\mathrm{e}^{-\phi}$ avec ϕ plurisousharmonique, alors, si B_θ est la boule unité de l'interpolé $[X,Y]_\theta$ entre X et Y en $\theta\in[0,1]$, la fonction $\theta\to\mu(B_\theta)$ est log-concave et donc en particulier

$$\mu(B_\theta)\geqslant\mu(B_X)^{1-\theta}\mu(B_Y)^\theta.$$

En interpolant entre X et $\overline{X}^*$ (dual conjugué de X), on obtient en $\theta=1/2$ l'espace euclidien (ou plutôt hermitien) de boule unité $D_{2n}=\{w\in\mathbb{C}^n;\ |w|\leqslant 1\}$. Ainsi, pour toute boule K de $\mathbb{C}^n$ et $\overline{K^\circ}:=\{\overline{w},\ |\sum_{i=1}^n w_iw_i'|\leqslant 1\ \forall w'\in K\}$, on a,

$$\mu(K)\mu\big(\overline{K^\circ}\big)\leqslant\mu(D_{2n})^2.$$

Comme conséquence on obtient donc l'inégalité de Santalo (1) pour les boules de $\mathbb{C}^n$ mais aussi, par exemple, l'inégalité $\gamma_{2n}(K)\gamma_{2n}(K^\circ)\leqslant\gamma_{2n}(D_{2n})^2$, où γ_{2n} est la mesure gaussienne sur $\mathbb{R}^{2n}$ (ce qui, plus généralement, est vrai pour les corps convexes symétriques de $\mathbb{R}^n$ d'après la méthode de Meyer et Pajor [4]). Il serait intérересsant de savoir si le résultat reste vrai sur $\mathbb{R}^n$ pour une mesure log-convave paire générale.

1. Introduction

Let K be a symmetric convex body of $\mathbb{R}^n$, i.e. the unit ball for some norm on $\mathbb{R}^n$, and K° its polar body, i.e. the unit ball of the dual space. Santaló's inequality [6] asserts that

$$\mathrm{vol}(K)\,\mathrm{vol}\big(K^\circ\big)\leqslant\mathrm{vol}(D_n)^2, \tag{2}$$

where vol$(\cdot)$ denotes the Lebesgue measure and $D_n \subset \mathbb{R}^n$ the standard Euclidean ball (*see* [4] for a modern proof based on Steiner's symmetrization). It is known that inequality (2) is naturally linked to complex interpolation. Indeed, complex interpolation between a space and its dual gives, "in the middle" (at $1/2$) the Euclidean space. Thus Santalo's inequality would follow from the log-concavity of the volume of the unit balls of the interpolated spaces. However, this approach was not carried to its end because such log-concavity property remained unknown. In fact, it was expected that the role played by convexity would be played, in this complex setting, by plurisubharmonic functions. But the important and missing tool was a complex version of Prékopa's theorem [5] for plurisubharmonic functions. Such a version was recently obtained by Berndtsson [2]. In the present note we show how Berndtsson's inequality can be used to obtain volume estimates, such as Santaló's inequality, for balls of $\mathbb{C}^n$. We stress that the idea of using complex interpolation in this setting is not ours. We are indebted to Bernard Maurey and Alain Pajor for bringing our attention on Berndtsson's paper [2] and for suggesting a possible application to Santaló type inequalities.

We will work on $\mathbb{C}^n$. A *ball* of $\mathbb{C}^n$ is a unit ball for some norm on $\mathbb{C}^n$. Equivalently, a ball is a convex compact set with non-empty interior, invariant under the action $\mathbb{C}^n \times \mathbb{R} \ni (w, \theta) \to \mathrm{e}^{\mathrm{i}\theta} w$. Thus a ball of $\mathbb{C}^n$ is a symmetric convex body of $\mathbb{R}^{2n}$ while the converse is false in general. In accordance, we will denote by D_{2n} the usual Euclidean (or rather Hermitian) ball

$$D_{2n} := \left\{ w \in \mathbb{C}^n;\ |w|^2 := \sum_{i=1}^{n} |w_i|^2 \leqslant 1 \right\}.$$

We will work with finite dimensional complex normed spaces $X = (\mathbb{C}^n, \|\cdot\|_X)$ where $\|\cdot\|_X$ is some norm on $\mathbb{C}^n$. The ball B_X is then defined by by $B_X := \{w \in \mathbb{C}^n;\ \|w\|_X \leqslant 1\}$. We will denote by $\langle\ ,\ \rangle$ the (bilinear) duality bracket: for $w^1, w^2 \in \mathbb{C}^n$, $\langle w^1, w^2\rangle := \sum_{k=1}^n w_k^1 w_k^2$, so that the dual of X is $X^* = (\mathbb{C}^n, \|\cdot\|_{X^*})$ where

$$\|w\|_{X^*} := \sup_{\|x\|_X \leqslant 1} \left|\langle w, x\rangle\right|.$$

The polar body B_X° of B_X is then defined by $B_X^\circ := B_{X^*}$.

For given (finite) dimensional Banach spaces $X = (\mathbb{C}^n, \|\cdot\|_X)$ and $Y = (\mathbb{C}^n, \|\cdot\|_Y)$ denote by $[X, Y]_z = (\mathbb{C}^n, \|\cdot\|_z)$ the complex interpolation between X and Y at $z \in C := \{z \in \mathbb{C} \mid 0 \leqslant \Re(z) \leqslant 1\}$. In Section 2 we briefly recall the definition of complex interpolation. In order to apply Berndtsson's result we need to observe that the function

$$\begin{aligned} \mathbb{C}^n \times C &\longrightarrow \mathbb{R}, \\ (w, z) &\longrightarrow \|w\|_z \end{aligned}$$

is plurisubharmonic. Then in Section 3 we reproduce Santaló's inequality (for a larger class of measures) for balls of $\mathbb{C}^n$. Of course, Santaló's inequality for balls of $\mathbb{C}^n$ is nothing but a particular case of Santaló's inequality on $\mathbb{R}^{2n}$. Thus the interesting points in this Note are the interpolation based approach together with the extension to other measures.

2. Plurisubharmonicity properties of the interpolated norms

A real-valued function defined on some domain of $\mathbb{C}^n$ is said to be plurisubharmonic if its restriction to every affine complex line is subharmonic. Plurisubharmonicity, as well as harmonicity, is a *local* property. If ϕ is plurisubharmonic and $\lambda \in \mathbb{R}$, the set $\{w \in \mathbb{C}^n;\ \phi(w) < \lambda\}$ is a pseudo-convex domain, and this is more or less the definition of a pseudo-convex domain (one can consult [3] for details).

We now briefly recall the definition of complex interpolation (*see* [1] for a detailed presentation). We then show that the interpolated norms satisfies some plurisubharmonic properties.

As before, set $C := \{z \in \mathbb{C} \mid \Re(z) \in [0, 1]\}$ and introduce the space

$$\mathcal{F} := \Big\{ f : C \longrightarrow \mathbb{C}^n \mid f \text{ bounded continuous on } C,\ \text{holomorphic on } \Re(z) \in (0,1) \text{ and}$$
$$\lim_{t \to \pm\infty} f(\alpha + \mathrm{i}t) = 0 \text{ for } \alpha = 0 \text{ or } 1 \Big\}.$$

For a given $Z = (\mathbb{C}^n, \|\cdot\|_Z)$ and $\alpha = 0$ or 1 define for $f \in \mathcal{F}$,

$$N_Z^\alpha(f) := \sup_{x \in \mathbb{R}} \big\| f(\alpha + \mathrm{i}x) \big\|_Z.$$

Let $X = (X, \|\cdot\|_X)$ and $Y = (X, \|\cdot\|_Y)$ be two n-dimensional complex normed spaces. Then for $z \in C$, the space $[X, Y]_z = (\mathbb{C}^n, \|\cdot\|_z)$ is defined by

$$\|w\|_z := \inf_{f \in \mathcal{F},\ f(z) = w} \max\big(N_X^0(f), N_Y^1(f)\big).$$

Note that $\|\cdot\|_z = \|\cdot\|_{\Re(z)}$ and so $[X, Y]_z = [X, Y]_\theta$ where $\theta = \Re(z) \in [0,1]$. Recall also that, since we work in finite dimension, $[X, Y]_z^* = [X^*, Y^*]_z$. As a consequence of this observation we emphasize the following result

LEMMA 2.1. – *Fix $X = (\mathbb{C}^n, \|\cdot\|_X)$, $Y = (\mathbb{C}^n, \|\cdot\|_Y)$ and introduce*

$$B_N\big(X^*, Y^*\big) := \big\{ f \in \mathcal{F} \mid N_{X^*}^0(f) \leqslant 1 \text{ and } N_{Y^*}^1(f) \leqslant 1 \big\}. \tag{3}$$

Then, for every $z \in C$ and every $w \in \mathbb{C}^n$, we have

$$\|w\|_z = \sup_{f \in B_N(X^*, Y^*)} \big| \big\langle f(z), w \big\rangle \big|.$$

With this observation in hand, it is easy to check that:

PROPOSITION 2.2 (Plurisubharmonocity of the interpolated norms). – *Fix $X = (\mathbb{C}^n, \|\cdot\|_X)$ and $Y = (\mathbb{C}^n, \|\cdot\|_Y)$ and denote as before by $[X, Y]_z = (\mathbb{C}^n, \|\cdot\|_z)$ the interpolated space at $z \in C$. Then the function*

$$\mathbb{C}^n \times C \longrightarrow \mathbb{R},$$
$$(w, z) \longrightarrow \|w\|_z$$

is plurisubharmonic.

Proof. – Fix $w^1, w^2 \in \mathbb{C}^n$ and $z_1, z_2 \in \mathbb{C}$ such that $\Re(z_1 + \mathrm{e}^{\mathrm{i}\theta} z_2) \in (0,1)$ for every $\theta \in \mathbb{R}$. We have to prove that

$$\big\| w^1 \big\|_{z_1} \leqslant \frac{1}{2\pi} \int_0^{2\pi} \big\| w^1 + \mathrm{e}^{\mathrm{i}\theta} w^2 \big\|_{z_1 + \mathrm{e}^{\mathrm{i}\theta} z_2} \,\mathrm{d}\theta. \tag{4}$$

For $f \in B_N(X^*, Y^*)$ (3) we have

$$\frac{1}{2\pi} \int_0^{2\pi} \big\| w^1 + \mathrm{e}^{\mathrm{i}\theta} w^2 \big\|_{z_1 + \mathrm{e}^{\mathrm{i}\theta} z_2} \,\mathrm{d}\theta \geqslant \frac{1}{2\pi} \int_0^{2\pi} \big| \big\langle f\big(z_1 + \mathrm{e}^{\mathrm{i}\theta} z_2\big), w^1 + \mathrm{e}^{\mathrm{i}\theta} w^2 \big\rangle \big| \,\mathrm{d}\theta$$
$$\geqslant \big| \big\langle f(z_1), w^1 \big\rangle \big|, \tag{5}$$

where we used Lemma 2.1 and the fact that the function $\xi \to \langle f(z_1 + \xi z_2), w^1 + \xi w^2 \rangle$ is holomorphic (where defined). Since (5) holds for every $f \in B_N(X^*, Y^*)$, Lemma 2.1 gives (4) and completes the proof of the proposition. □

Although it is not needed here, we include a short proof of the following result.

PROPOSITION 2.3 (Log-plurisubharmoncity of the interpolated norms). – *Fix $X = (\mathbb{C}^n, \|\cdot\|_X)$ and $Y = (\mathbb{C}^n, \|\cdot\|_Y)$ and denote as before by $[X, Y]_z = (\mathbb{C}^n, \|\cdot\|_z)$ the interpolated space at $z \in C$. Then the function*

$$\mathbb{C}^n \times C \longrightarrow \mathbb{R} \cup \{-\infty\},$$
$$(w, z) \longrightarrow \log \|w\|_z$$

is plurisubharmonic.

Proof. – Fix $w^1, w^2 \in \mathbb{C}^n$, $z_1, z_2 \in \mathbb{C}$ and a domain $\Omega \subset \mathbb{C}$ such that $\Re(z_1 + \xi z_2) \in (0, 1)$ for every $\xi \in \Omega$. Let F be a function holomorphic in a neighborhood of Ω such that

$$\log\left\|w^1 + \xi w^2\right\|_{z_1+\xi z_2} \leqslant \Re\bigl(F(\xi)\bigr) \quad \forall \xi \in \partial\Omega. \tag{6}$$

We want to prove that this implies the same inequality on Ω. Since (6) is equivalent to

$$\left\|\mathrm{e}^{-F(\xi)}\bigl(w^1 + \xi w^2\bigr)\right\|_{z_1+\xi z_2} \leqslant 1 \quad \forall \xi \in \partial\Omega,$$

we have by Lemma 2.1:

$$\forall f \in B_N\bigl(X^*, Y^*\bigr)\ \forall \xi \in \partial\Omega \quad \left|\bigl\langle f(z_1 + \xi z_2), \mathrm{e}^{-F(\xi)}\bigl(w^1 + \xi w^2\bigr)\bigr\rangle\right| \leqslant 1.$$

For a fixed $f \in B_N(X^*, Y^*)$ the function $\xi \to \langle f(z_1 + \xi z_2), \mathrm{e}^{-F(\xi)}(w^1 + \xi w^2)\rangle$ is holomorphic and therefore

$$\forall f \in B_N\bigl(X^*, Y^*\bigr)\ \forall \xi \in \Omega \quad \left|\bigl\langle f(z_1 + \xi z_2), \mathrm{e}^{-F(\xi)}\bigl(w^1 + \xi w^2\bigr)\bigr\rangle\right| \leqslant 1.$$

Again by Lemma 2.1, this implies

$$\left\|\mathrm{e}^{-F(\xi)}\bigl(w^1 + \xi w^2\bigr)\right\|_{z_1+\xi z_2} \leqslant 1 \quad \forall \xi \in \Omega.$$

This is equivalent to $\log\|w^1 + \xi w^2\|_{z_1+\xi z_2} \leqslant \Re(F(\xi))$ for every $\xi \in \Omega$. □

3. Volume estimates for balls of $\mathbb{C}^n$

In this section μ will be a measure on $\mathbb{C}^n$ verifying

$$\text{(H)} \quad \begin{cases} \mathrm{d}\mu(w) = \mathrm{e}^{-\varphi(w)}\,\mathrm{d}\,\mathrm{vol}(w), \\ \varphi : U \to \mathbb{R} \text{ is plurisubharmonic on a pseudo-convex domain } U \subset \mathbb{C}^n, \\ \varphi \text{ and } U \text{ are invariant under the action } (w,\theta) \to \mathrm{e}^{\mathrm{i}\theta} w, \text{ and } 0 \in U. \end{cases}$$

We have in mind the following two situations: $\mu = \mathrm{vol}$ obtained by taking $\varphi = 0$ on $\mathbb{C}^n$, and the Gaussian measure $\mu = \gamma_{2n}$ obtained by taking $\varphi(w) = |w|^2/2 + n\log(2\pi)$ on $\mathbb{C}^n$.

As we mentioned, the central ingredient here will be Berndtsson's generalization of Prékopa's inequality asserting that marginals of a $(-\log)$-plurisubharmonic measure are $(-\log)$-plurisubharmonic. As noted by Berndtsson, an invariance property is required for the result to be true. The proof of Berndtsson relies on a surprising use of Hörmander's L^2 estimate for the $\overline{\partial}$-operator. The result we need reads as:

THEOREM 3.1 (Berndtsson [2]). – *Let Ω be a pseudo-convex domain of $\mathbb{C}^n \times \mathbb{C}$ and $\phi : \Omega \to \mathbb{R} \cup \{-\infty\}$ be a plurisubharmonic function. Set $P := \{z \in \mathbb{C};\ \exists w \in \mathbb{C}^n;\ (w, z) \in \Omega\}$ and for $z \in P$, $\Omega(z) := \{w \in \mathbb{C}^n;\ (w, z) \in \Omega\}$. For $z \in P$, assume that $0 \in \Omega(z)$ and that, for every $w \in \Omega(z)$ and $\theta \in \mathbb{R}$, $\mathrm{e}^{\mathrm{i}\theta} w \in \Omega(z)$ and $\phi(\mathrm{e}^{\mathrm{i}\theta} w, z) = \phi(w, z)$. Then the function Φ defined on P by*

$$\mathrm{e}^{-\Phi(z)} = \int_{\Omega(z)} \mathrm{e}^{-\phi(w,z)}\,\mathrm{d}\,\mathrm{vol}(w)$$

is subharmonic.

We now combine Proposition 2.2 and Theorem 3.1.

THEOREM 3.2 (Log-concavity of the measure of balls along interpolation). – *Let μ be a measure verifying* (H). *Fix $X = (\mathbb{C}^n, \|\cdot\|_X)$ and $Y = (\mathbb{C}^n, \|\cdot\|_Y)$ and denote by $[X, Y]_\theta = (\mathbb{C}^n, \|\cdot\|_\theta)$ the complex interpolated space between X and Y at $\theta \in [0, 1]$ with unit ball B_θ. Then the function $\theta \to \mu(B_\theta)$ is log-*

concave. In particular:

$$\mu(B_\theta) \geqslant \mu(B_X)^{1-\theta}\mu(B_Y)^\theta.$$

Proof. – Denote by $\widetilde{C} := \{z \in \mathbb{C} \mid \Re(z) \in (0,1)\}$ the interior of the strip C and introduce $\Omega := \{(w,z) \in \mathbb{C}^n \times \widetilde{C};\ \|w\|_z < 1\}$. Because of Proposition 2.2, the set Ω is a pseudo-convex domain of $\mathbb{C}^n \times \mathbb{C}$. For $(w,z) \in \Omega$ introduce $\phi(w,z) := \varphi(w)$. Note that Ω and ϕ satisfy all the assumptions required in Theorem 3.1. Furthermore $\Omega(z)$ is the open unit ball of $[X,Y]_z$ and thus the function

$$z \longrightarrow -\log \int_{B_z} \mathrm{e}^{-\varphi(w)}\,\mathrm{d}\,\mathrm{vol}(w) = -\log\mu(B_z)$$

is subharmonic on $\widetilde{C}$. But, this function depends only on $\Re(z)$ and thus subharmonicity reduces to the convexity on $(0,1)$ of the function $\theta \to -\log\mu(B_\theta)$. □

Note that the theorem in particular gives that $\mathrm{vol}(B_\theta) \geqslant \mathrm{vol}(B_X)^{1-\theta}\,\mathrm{vol}(B_Y)^\theta$.

We now turn to self dual estimates. If $X = (\mathbb{C}^n, \|\cdot\|_X)$ is an n-dimensional complex normed space, we will denote by $\overline{X}$ the space whose norm is given by $\|w\|_{\overline{X}} := \|\overline{w}\|_X$ for every $w \in \mathbb{C}^n$. The unit ball $B_{\overline{X}}$ of $\overline{X}$ verifies

$$B_{\overline{X}} = \overline{B_X} := \{\overline{w};\ w \in B_X\}.$$

It is easily checked that $(\overline{X})^* = \overline{X^*}$. We denote by ℓ_2^{2n} the standard Euclidean (or rather Hermitian) space $\ell_2^{2n} := (\mathbb{C}^n, |\cdot|)$ with unit ball D_{2n}. One has $\ell_2^{2n} = \overline{\ell_2^{2n}} = (\ell_2^{2n})^*$. It is classical that for $X = (\mathbb{C}^n, \|\cdot\|)$ one has

$$\big[X, \overline{X}^*\big]_{1/2} = \ell_2^{2n}.$$

Therefore, as a consequence of Theorem 3.2 we have

COROLLARY 3.3. – *Let μ be a measure verifying* (H) *and let K be a ball of $\mathbb{C}^n$. One has*

$$\mu(K)\mu\big(\overline{K^\circ}\big) \leqslant \mu(D_{2n})^2.$$

In particular one has $\mathrm{vol}(K)\,\mathrm{vol}(K^\circ) \leqslant \mathrm{vol}(D_{2n})^2$.

One has also $\gamma_{2n}(K)\gamma_{2n}(K^\circ) \leqslant \gamma_{2n}(D_{2n})^2$. It follows from Meyer and Pajor's method [4] that this inequality is more generally true on $\mathbb{R}^n$ for symmetric convex bodies. It would be interesting to know if the result also holds for general even log-concave measures on $\mathbb{R}^n$. For instance, for balls K, L of $\mathbb{C}^n$ (with $\overline{L} = L$) one has, as a consequence of Corollary 3.3 (with $\mathrm{d}\mu = 1_L\,\mathrm{d}\,\mathrm{vol}$),

$$\mathrm{vol}(K \cap L)\,\mathrm{vol}\big(K^\circ \cap L\big) \leqslant \mathrm{vol}(D_{2n} \cap L)^2.$$

Does this inequality hold on $\mathbb{R}^n$ for symmetric convex bodies?

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