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Équations aux dérivées partielles/*Partial Differential Equations*

A comparison result related to Gauss measure

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Abstract In this paper we prove a comparison result for weak solutions to linear elliptic problems of the type

$$-(a_{ij}(x)u_{x_i})_{x_j} = f(x)\varphi(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is an open set of $\mathbb{R}^n$ $(n \geqslant 2)$, $\varphi(x) = (2\pi)^{-n/2}\exp(-|x|^2/2)$, $a_{ij}(x)$ are measurable functions such that $a_{ij}(x)\xi_i\xi_j \geqslant \varphi(x)|\xi|^2$ a.e. $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $f(x)$ is a measurable function taken in order to guarantee the existence of a solution $u \in \mathrm{H}_0^1(\varphi, \Omega)$ of (1.1). We use the notion of rearrangement related to Gauss measure to compare $u(x)$ with the solution of a problem of the same type, whose data are defined in a half-space and depend only on one variable. ***To cite this article: M.F. Betta et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 451–456.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Un résultat de comparaison relatif à la mesure de Gauss

Résumé Dans cette note on démontre un résultat de comparaison pour les solutions faibles de problèmes elliptiques linéaires du type

$$-(a_{ij}(x)u_{x_i})_{x_j} = f(x)\varphi(x) \quad \text{dans } \Omega, \qquad u = 0 \quad \text{sur } \partial\Omega,$$

où Ω est un ouvert de $\mathbb{R}^n$ $(n \geqslant 2)$, $\varphi(x) = (2\pi)^{-n/2}\exp(-|x|^2/2)$, $a_{ij}(x)$ sont des fonctions mesurables telles que $a_{ij}(x)\xi_i\xi_j \geqslant \varphi(x)|\xi|^2$ p.p. $x \in \Omega$, $\xi \in \mathbb{R}^n$ et $f(x)$ est une fonction mesurable telle qu'il existe une solution u de (0.1), dans $\mathrm{H}_0^1(\varphi, \Omega)$. On utilise la notion de rearrangement relatif à la mesure de Gauss pour comparer $u(x)$ avec la solution d'un problème du même type, dont les données sont définies dans un demi plan et dépendent d'une variable seulement. ***Pour citer cet article : M.F. Betta et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 451–456.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

On considère le problème de Dirichlet suivant

$$-\big(a_{ij}(x)u_{x_i}\big)_{x_j} = f(x)\varphi(x) \quad \text{dans } \Omega, \qquad u = 0 \quad \text{sur } \partial\Omega, \tag{0.1}$$

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où Ω est un ouvert de $\mathbb{R}^n$ $(n \geqslant 2)$, $\varphi(x) = (2\pi)^{-n/2}\exp(-|x|^2/2)$, $a_{ij}(x)$ sont des fonctions mesurables sur Ω, telles que $a_{ij}(x)\xi_i\xi_j \geqslant \varphi(x)|\xi|^2$ p.p. $x \in \Omega$, $\forall \xi \in \mathbb{R}^n$, et $f(x)$ est une fonction mesurable telle qu'une solution $u \in \mathrm{H}_0^1(\varphi, \Omega)$ de (0.1) existe. On remarque que l'équation (0.1) est dégénérée si Ω n'est pas borné. Le but est d'obtenir des estimations a priori pour les solutions faibles de (0.1). De telles estimations peuvent souvent être obtenues en comparant le problème d'origine avec un problème plus simple, « symétrisé » dans une boule ayant la même mesure de Lebesgue que Ω. Cependant, l'utilisation de la mesure de Lebesgue n'est pas appropriée dans notre cas à cause de la dégénérescence de l'opérateur et du fait que Ω peut étre de mesure de Lebesgue infinie. Aussi on utilisera le rearrangement relatif à la mesure de Gauss $\gamma(\mathrm{d}x) = (2\pi)^{-n/2}\exp(-|x|^2/2)\,\mathrm{d}x$.

THEOREM 0.1. – *Soit $\lambda \in \mathbb{R}$ défini par $\gamma(\{x_1 > \lambda\}) = \gamma(\Omega)$. On note alors $\Omega^\star$ le demi plan $\{x_1 > \lambda\}$ et $v^\star = v^\star(x_1)(\Omega^\star \to \mathbb{R})$ le réarrangement de $v(\Omega \to \mathbb{R})$ relatif à la mesure $\gamma(\mathrm{d}x)$. Pour $f(\Omega \to \mathbb{R})$, la condition*

$$\int_\lambda^{+\infty} \exp\left(\frac{\tau^2}{2}\right)\left(\int_\tau^{+\infty} f^\star(\sigma)\exp\left(-\frac{\sigma^2}{2}\right)\mathrm{d}\sigma\right)^2 \mathrm{d}\tau < +\infty \tag{0.2}$$

est nécessaire et suffisante pour que le problème

$$-\big(\varphi(x)w_{x_1}\big)_{x_1} = f^\star(x_1)\varphi(x) \quad \textit{dans } \Omega^\star, \qquad w = 0 \quad \textit{sur } \partial\Omega^\star \tag{0.3}$$

admet une solution ; celle-ci est alors de la forme $w = w(x_1)$ (définie explicitement). De plus, dès que u est solution de (0.1)*, on a*

$$u^\star(x_1) \leqslant w(x) \quad \textit{p.p. } x \in \Omega^\star, \tag{0.4}$$

$$\int_\Omega |\nabla u|^q \varphi(x)\,\mathrm{d}x \leqslant \int_{\Omega^\star} |\nabla w|^q \varphi(x)\,\mathrm{d}x, \quad 0 < q \leqslant 2, \tag{0.5}$$

où $u^\star$ est le réarrangement de u, relatif à la mesure de Gauss, et w est la solution du problème (0.3).

1. Introduction

Let us consider the following Dirichlet problem

$$-\big(a_{ij}(x)u_{x_i}\big)_{x_j} = f(x)\varphi(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^n$ $(n \geqslant 2)$, $\varphi(x) = (2\pi)^{-n/2}\exp(-|x|^2/2)$, $a_{ij}(x)$ are measurable functions on Ω such that

$$a_{ij}(x)\xi_i\xi_j \geqslant \varphi(x)|\xi|^2 \quad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^n, \tag{1.2}$$

and f is a measurable function, taken in order to guarantee the existence of a solution u in the weighted Sobolev space $\mathrm{H}_0^1(\varphi, \Omega)$.[1] We emphasize that Eq. (1.1) degenerates when Ω is unbounded. Our aim is to obtain a priori estimates for weak solutions of problem (1.1).

It is well known that such estimates can often be obtained by comparing the original problem with a simpler "symmetrized" one in a ball, which has the same Lebesgue measure as Ω. The estimates are then proved by using the classical isoperimetric inequality in $\mathbb{R}^n$ and some further inequalities between integrals of a given function and its Schwarz symmetrization (see, for instance, [1,2,8] and [10]). However, Schwarz symmetrization is not an appropriate tool in our case because of the degeneracy of the operator and since the set Ω can have infinite Lebesgue measure.

Let us consider the Gauss measure on $\mathbb{R}^n$ defined by $\gamma(\mathrm{d}x) = \varphi(x)\,\mathrm{d}x$.

We compare (1.1) with an analogous problem in a half-space, which has the same n-dimensional Gauss measure as Ω. In other words, the standard Schwarz symmetrization is replaced by a rearrangement with respect to Gauss measure on $\mathbb{R}^n$, defined by $\gamma(\mathrm{d}x) = \varphi(x)\,\mathrm{d}x$. Notice that we have $\gamma(\mathbb{R}^n) = 1$.

In Section 2 we will give the definition of such a rearrangement. Here we mention that the rearrangement with respect to Gauss measure is a mapping, which transforms a domain Ω into the half-space $\Omega^\star = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$, $\lambda \in \mathbb{R}$, with $\gamma(\Omega) = \gamma(\Omega^\star)$. Moreover it transforms a measurable function u defined on Ω into another function $u^\star$, which is defined on $\Omega^\star$, which depends only on the variable x_1 and such that its level sets $\{u^\star > t\}$ are half-spaces $\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > \mu\}$, $\mu \in \mathbb{R}$, having the same Gauss measure as the corresponding level sets of $|u|$, $\{|u| > t\}$.

Let us now describe in detail the comparison result we are interested in. We consider the class of problems (1.1) letting Ω vary in the class of open subsets of $\mathbb{R}^n$ having fixed Gauss measure and letting f vary in a set of functions that ensure the existence of the solution of (1.1) and have fixed rearrangement with respect to Gauss measure. Then we ask for which domain Ω and right-hand side f, various Sobolev norms of u in $\mathrm{H}^1_0(\varphi, \Omega)$ are as large as possible. It turns out that the optimum of these norms is achieved for the half-space $\Omega^\star$ and for the right-hand side $f^\star$ in (1.1). To this aim, we prove that the pointwise comparison holds true (*see* Section 3)

$$u^\star(x) = u^\star(x_1) \leqslant w(x) = w(x_1) \quad \text{for a.e. } x = (x_1, x_2, \ldots, x_n) \in \Omega^\star,$$

where $u(x)$ is the solution of the problem (1.1) and $w(x) = w^\star(x_1)$ is the solution of the following problem, defined in $\Omega^\star$ and whose data depend only on the first variable:

$$-\big(\varphi(x) w_{x_1}\big)_{x_1} = f^\star(x_1)\varphi(x) \quad \text{in } \Omega^\star, \qquad w = 0 \quad \text{on } \partial\Omega^\star. \tag{1.3}$$

The method of our proof is an adaptation of the classical method of level sets, where the isoperimetric inequality in Gaussian space (*see* Section 2) plays a central rule.

Comparison results in this order of idea are contained in [3,4].

2. Preliminaries

Let $\gamma(\mathrm{d}x)$ be the n-dimensional Gauss measure on $\mathbb{R}^n$ defined by $\gamma(\mathrm{d}x) = (2\pi)^{-n/2} \exp(-|x|^2/2)\,\mathrm{d}x$, $x \in \mathbb{R}^n$, normalized by $\gamma(\mathbb{R}^n) = 1$. If E is a $(n-1)$-rectifiable set, we define the perimeter of E by

$$P(E) = (2\pi)^{-n/2} \int_{\partial E} \exp\left(-\frac{|x|^2}{2}\right) \mathcal{H}_{n-1}(\mathrm{d}x),$$

where $\mathcal{H}_{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. We denote by $E^\star$ the half-space defined by $E^\star = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$, for $\lambda \in \mathbb{R}$ such that $\gamma(E) = \gamma(E^\star)$. $E^\star$ is called the *rearrangement of E* (*with respect to Gauss measure*). In [5] the following isoperimetric inequality is proved

$$P(E) \geqslant P\big(E^\star\big). \tag{2.1}$$

Next, let u be a measurable function defined in a subset Ω of $\mathbb{R}^n$. The *distribution function of u*, denoted by μ, is the map from $[0, +\infty[$ into $[0, 1]$ defined by:

$$\mu(t) = \gamma\big(\big\{x \in \Omega : \big|u(x)\big| > t\big\}\big).$$

The function μ is decreasing and right-continuous. The *decreasing rearrangement of u* (*with respect to Gauss measure*), is the decreasing, right-continuous function $u^* :]0, 1] \to [0, +\infty[$, defined by

$$u^*(s) = \inf\big\{t \geqslant 0 : \mu(t) \leqslant s\big\}, \quad 0 < s \leqslant 1.$$

Setting

$$\Phi(\tau) := \gamma\big(\big\{x \in \mathbb{R}^n : x_1 > \tau\big\}\big) = \frac{1}{\sqrt{2\pi}} \int_\tau^{+\infty} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t, \quad \tau \in \mathbb{R}, \tag{2.2}$$

we define the *rearrangement of* u (*with respect to Gauss measure*) as the function $u^\star : \Omega^\star \to [0, +\infty[$ defined by

$$u^\star(x) \equiv u^\star(x_1) = u^*\big(\Phi(x_1)\big).$$

More details on rearrangements with respect to positive measures are contained, for example, in [9] and [7]; we just recall that the following Hardy–Littlewood type inequality holds

$$\int_\Omega \big|f(x)g(x)\big|\gamma(\mathrm{d}x) \leqslant \int_{\Omega^\star} f^\star(x)g^\star(x)\gamma(\mathrm{d}x) = \int_0^{\gamma(\Omega)} f^*(s)g^*(s)\,\mathrm{d}s. \tag{2.3}$$

3. Main result

In this Note we assume that there exists a solution of the problem (1.1), namely a function $u \in \mathrm{H}_0^1(\varphi, \Omega)$, which verifies

$$\int_\Omega a_{ij} u_{x_i} \phi_{x_j}\,\mathrm{d}x = \int_\Omega f\varphi\phi\,\mathrm{d}x, \quad \forall \phi \in \mathrm{H}_0^1(\varphi, \Omega). \tag{3.1}$$

Conditions which garantee the existence of a such solution can be found, for example, in [11].

THEOREM 3.1. – *Consider the rearrangement transforming* Ω *into* $\Omega^\star = \{x_1 > \lambda\}$ *and* $v : \Omega \to \mathbb{R}$ *into* $v^\star : \Omega^\star \to \mathbb{R}$, *as defined above. For* f *measurable function, we introduce the following condition*

$$\int_\lambda^{+\infty} \exp\left(\frac{\tau^2}{2}\right)\left(\int_\tau^{+\infty} f^\star(\sigma)\exp\left(-\frac{\sigma^2}{2}\right)\mathrm{d}\sigma\right)^2 \mathrm{d}\tau < +\infty. \tag{3.2}$$

The function w *defined by* $w(x) = w^\star(x_1) = \int_\lambda^{x_1} \exp(\tau^2/2) \int_\tau^{+\infty} f^\star(\sigma)\exp(-\sigma^2/2)\,\mathrm{d}\sigma\,\mathrm{d}\tau$ *is the unique solution of the problem* (1.3) *if, and only if,* (3.2) *holds true. Moreover, as soon as* u *solves problem* (1.1), *we have*

$$u^\star(x_1) \leqslant w(x) \quad \text{for a.e. } x \in \Omega^\star, \tag{3.3}$$

$$\int_\Omega |\nabla u|^q \varphi(x)\,\mathrm{d}x \leqslant \int_{\Omega^\star} |\nabla w|^q \varphi(x)\,\mathrm{d}x \quad \text{for all } 0 < q \leqslant 2. \tag{3.4}$$

Remark 1. – Condition (3.2) is satisfied for a wide class of functions, for instance for functions f satisfying $f^\star(\tau) \leqslant C\exp(\tau^2/4)(1+|\tau|)^{1/2-\varepsilon}$, for every $\tau \geqslant \lambda$, for some constants $C > 0$, $\varepsilon > 0$.

Proof. – One easily checks that $w = w(x_1)$ is the unique solution of (1.3) and that

$$\int_{\Omega^\star} |\nabla w|^2 \varphi(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_\lambda^{+\infty} \exp\left(\frac{r^2}{2}\right)\left(\int_r^{+\infty} f^\star(\sigma)\exp\left(-\frac{\sigma^2}{2}\right)\mathrm{d}\sigma\right)^2 \mathrm{d}r < +\infty,$$

under assumption (3.2). Besides the above equality obviously implies the necessity of (3.2). Our proof of (3.3) and (3.4) is the following one. Let $t \in [0, \operatorname{ess\,sup}|u|[$ and $h > 0$. We choose as test function in (3.1)

$$\phi_h = \begin{cases} \operatorname{sign} u & \text{if } |u| > t + h, \\ \dfrac{u - t\operatorname{sign} u}{h} & \text{if } t < |u| \leqslant t + h, \\ 0 & \text{otherwise.} \end{cases}$$

Then using (1.2) and letting h go to zero, we get for a.e. $t \in [0, +\infty[$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t} |\nabla u|^2 \varphi(x)\,\mathrm{d}x \leqslant \int_{|u|>t} \big|f(x)\big|\varphi(x)\,\mathrm{d}x. \tag{3.5}$$

Applying Cauchy–Schwarz inequality to the difference quotient, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t} |\nabla u|\varphi(x)\,\mathrm{d}x \leqslant \left(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t} |\nabla u|^2\varphi(x)\,\mathrm{d}x\right)^{1/2}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t} \varphi(x)\,\mathrm{d}x\right)^{1/2}. \tag{3.6}$$

On the other hand, coarea formula (*see* [6]) and isoperimetric inequality with respect to the Gauss measure (2.1) give

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}|\nabla u|\varphi(x)\,\mathrm{d}x=\int_{\partial\{|u|>t\}}\varphi(x)\,\mathcal{H}_{n-1}(\mathrm{d}x)\geqslant\int_{\partial\{|u|>t\}^\star}\varphi(x)\mathcal{H}_{n-1}(\mathrm{d}x). \tag{3.7}$$

Then, by (3.5), using (3.6), (3.7) and Hardy inequality (2.3), we have

$$\frac{1}{-\mu'(t)}\left(\int_{\partial\{|u|>t\}^\star}\varphi(x)\mathcal{H}_{n-1}(\mathrm{d}x)\right)^2\leqslant\int_0^{\mu(t)}f^*(s)\,\mathrm{d}s. \tag{3.8}$$

We now recall that the set $\{u^\star>t\}=\{|u|>t\}^\star$ is the half-space $\{x_1>\tau\}$ such that $\mu(t)=\gamma(\{x_1>\tau\})=\Phi(\tau)$ (*see* (2.2)), that is $\tau=\Phi^{-1}(\mu(t))$. Then

$$\begin{aligned}\int_{\partial\{|u|>t\}^\star}\varphi(x)\mathcal{H}_{n-1}(\mathrm{d}x)&=\frac{1}{(2\pi)^{n/2}}\int_{x_1=\Phi^{-1}(\mu(t))}\exp\left(-\frac{|x|^2}{2}\right)\mathcal{H}_{n-1}(\mathrm{d}x)\\&=\frac{1}{\sqrt{2\pi}}\exp\left(-\Phi^{-1}\frac{(\mu(t))^2}{2}\right).\end{aligned} \tag{3.9}$$

Using (3.9) in (3.8) we have

$$1\leqslant 2\pi\exp\left[\left(\Phi^{-1}\left(\mu(t)\right)\right)^2\right]\left(-\mu'(t)\right)\int_0^{\mu(t)}f^*(s)\,\mathrm{d}s. \tag{3.10}$$

Integrating between 0 and t and putting $\mu(t)=s$, (3.10) becomes

$$u^*(s)\leqslant 2\pi\int_s^{\gamma(\Omega)}\exp\left(\left(\Phi^{-1}(\sigma)\right)^2\right)\left(\int_0^\sigma f^*(\tau)\,\mathrm{d}\tau\right)\mathrm{d}\sigma=w^*(s),\quad \text{a.e. } 0<s\leqslant\gamma(\Omega). \tag{3.11}$$

Now we take $s=\Phi(x_1)$ and we recall that $\tau=\Phi^{-1}(\mu(t))$; then (3.11) gives (3.3).

Let us prove now (3.4). Using Hölder inequality and (3.5) we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}|\nabla u|^q\varphi(x)\,\mathrm{d}x\leqslant\left(\int_{|u|>t}\left|f(x)\right|\varphi(x)\,\mathrm{d}x\right)^{q/2}\left(-\mu'(t)\right)^{1-q/2}.$$

Then from Hardy inequality and (3.10), we get

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}|\nabla u|^q\varphi(x)\,\mathrm{d}x&\leqslant\left(\int_0^{\mu(t)}f^*(s)\,\mathrm{d}s\right)^{q/2}\left(-\mu'(t)\right)^{1-q/2}\\&\leqslant(2\pi)^{q/2}\exp\left(\frac{q}{2}\left[\Phi^{-1}\left(\mu(t)\right)\right]^2\right)\left(\int_0^{\mu(t)}f^*(s)\,\mathrm{d}s\right)^q\left(-\mu'(t)\right).\end{aligned}$$

Integrating between 0 and $+\infty$, and choosing λ such that $\Omega^\star=\{x_1>\lambda\}$, we have

$$\begin{aligned}\int_\Omega|\nabla u|^q\varphi(x)\,\mathrm{d}x&\leqslant(2\pi)^{q/2}\int_0^{\gamma(\Omega)}\exp\left(\frac{q}{2}\left[\Phi^{-1}(s)\right]^2\right)\left(\int_0^s f^*(\tau)\,\mathrm{d}\tau\right)^q\mathrm{d}s\\&=\frac{1}{\sqrt{2\pi}}\int_\lambda^{+\infty}\exp\left(\frac{q-1}{2}r^2\right)\left(\int_r^{+\infty}f^\star(\sigma)\exp\left(-\frac{\sigma^2}{2}\right)\mathrm{d}\sigma\right)^q\mathrm{d}r,\end{aligned} \tag{3.12}$$

that is (3.4).

[1] We denote by $\mathrm{H}_0^1(\varphi,\Omega)$ the closure of $\mathrm{C}_0^\infty(\Omega)$ under the norm $\left(\int_\Omega|\nabla u(x)|^2\varphi(x)\,\mathrm{d}x\right)^{1/2}$.

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