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Équations aux dérivées partielles/*Partial Differential Equations*

On some conformally invariant fully nonlinear equations

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Abstract We outline proofs of our results in [7] on Liouville type theorems, Harnack type inequalities, and existence and compactness of solutions to some conformally invariant fully nonlinear elliptic equations of second order on locally conformally flat Riemannian manifolds. Details will appear in [7]. ***To cite this article: A. Li, Y.Y. Li, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 305–310.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur certaines équations completement nonlinéaires invariantes par transformation conforme

Résumé On présente des résultats de type Liouville, des inégalités de type Harnack ainsi que d'existence et de compacité de solutions pour certaines équations elliptiques du second ordre, complètement nonlinéaires, sur des variétés Riemanniennes localement conformément plates. Les démonstrations détaillées sont contenues dans [7]. ***Pour citer cet article : A. Li, Y.Y. Li, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 305–310.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit (M,g) une variété Riemannienne régulière, sans bord, de dimension n. On considère le tenseur symétrique

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{2(n-1)}g\right),$$

où Ric et R désignent respectivement le tenseur de Ricci et la courbure scalaire associés à g.

Pour $1 \leqslant k \leqslant n$ et $\mu \in \mathbb{R}^n$, on note $\sigma_k(\mu)$ la fonction symétrique d'ordre k et Γ_k la composante connexe de $\{\mu \in \mathbb{R}^n \mid \sigma_k(\mu) > 0\}$ contenant le cône positif $\{\mu \in \mathbb{R}^n \mid \mu_1, \ldots, \mu_n > 0\}$. On s'intéresse à l'équation de courbure

$$\sigma_k\big(\mu(A_g)\big) = 1, \quad \mu(A_g) \in \Gamma_k, \tag{1}$$

où $\mu(A_g)$ dénote les valeurs propres de A_g.

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Soit g_{flat} la métrique Euclidienne sur $\mathbb{R}^n$.

THÉORÈME 1. – *Pour $n \geqslant 3$ et $1 \leqslant k \leqslant n$, soit $u \in C^2(\mathbb{R}^n)$ une fonction positive et soit $g = u^{\frac{4}{n-2}} g_{\text{flat}}$ une solution de* (1) *sur $\mathbb{R}^n$. Alors u est de la forme*

$$u(x) = c(n,k)\left(\frac{a}{1+a^2|x-\bar{x}|^2}\right)^{(n-2)/2}, \quad \forall x \in \mathbb{R}^n,$$

pour un certain $a > 0$ et $\bar{x} \in \mathbb{R}^n$, où $c(n,k) = 2^{(n-2)/4}\binom{n}{k}^{(n-2)/4k}$.

THÉORÈME 2. – *Soit $n \geqslant 3$, $1 \leqslant k \leqslant n$, $R > 0$, et $B_{3R} \subset \mathbb{R}^n$ une boule de rayon $3R$. Soit $u \in C^2(B_{3R})$ une fonction positive et soit $g = u^{4/(n-2)} g_{\text{flat}}$ une solution de* (1) *dans B_{3R}. Alors*

$$\Bigl(\max_{\overline{B}_R} u\Bigr)\Bigl(\min_{\overline{B}_{2R}} u\Bigr) \leqslant C(n) R^{2-n}.$$

THÉORÈME 3. – *Soit $n \geqslant 3$ et $1 \leqslant k \leqslant n$. Soit (M, g_0) une variété Riemannienne compacte, régulière, sans bord, localement conformément plate, vérifiant $\mu(A_{g_0}) \in \Gamma_k$ sur M. Alors il existe une fonction u, positive et régulière sur M telle que $g = u^{4/(n-2)} g_0$ vérifie* (1) *sur M. De plus, si (M, g_0) n'est pas conformément difféomorphe à la n-sphère standard, alors toutes les solutions vérifient, pour tout m,*

$$\|u\|_{C^m(M,g_0)} + \bigl\|u^{-1}\bigr\|_{C^m(M,g_0)} \leqslant C,$$

où C dépend seulement de (M, g_0) et de m.

Conformally invariant fully nonlinear elliptic equations have attracted much attention recently, see, e.g., Viaclovsky [15–17], Chang, Gursky and Yang [3,4], and Guan and Wang [6].

Let (M, g) be an n-dimensional smooth Riemannian manifold without boundary, consider the symmetric tensor

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{2(n-1)} g\right),$$

where Ric and R denote respectively the Ricci tensor and the scalar curvature associated with g.

For $1 \leqslant k \leqslant n$, let

$$\sigma_k(\mu) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \mu_{i_1} \cdots \mu_{i_k}, \quad \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n,$$

denote the k-th symmetric function, and let Γ_k denote the connected component of $\{\mu \in \mathbb{R}^n \mid \sigma_k(\mu) > 0\}$ containing the positive cone $\{\mu \in \mathbb{R}^n \mid \mu_1, \ldots, \mu_n > 0\}$.

We are interested in the following curvature equation

$$\sigma_k\bigl(\mu(A_g)\bigr) = 1, \tag{2}$$

where $\mu(A_g)$ denotes the eigenvalues of A_g. In fact, we will only consider in this Note Eq. (2) together with

$$\mu(A_g) \in \Gamma_k. \tag{3}$$

Let $g = u^{4/(n-2)} g_{\text{flat}}$, where g_{flat} denotes the Euclidean metric on $\mathbb{R}^n$. Then

$$A_g = A^u,$$

where

$$A^u := -\frac{2}{n-2} u^{-(n+2)/(n-2)} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-2n/(n-2)} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-2n/(n-2)} |\nabla u|^2 I, \tag{4}$$

and I is the $n \times n$ identity matrix.

Eqs. (2) and (3) take the form

$$\sigma_k\big(\mu\big(A^u\big)\big)=1 \quad \text{on } \mathbb{R}^n, \tag{5}$$

and

$$\mu\big(A^u(x)\big)\in\Gamma_k, \quad \forall x\in\mathbb{R}^n. \tag{6}$$

For $k=1$, Eq. (5) becomes

$$-\Delta u=\frac{n-2}{2}u^{(n+2)/(n-2)} \quad \text{on } \mathbb{R}^n. \tag{7}$$

A Liouville type theorem of Caffarelli, Gidas and Spruck in [1] asserts that positive C^2 solutions of (7) are of the form

$$u(x)=(2n)^{(n-2)/4}\left(\frac{a}{1+a^2|x-\bar{x}|^2}\right)^{(n-2)/2},$$

where $a>0$ and $\bar{x}\in\mathbb{R}^n$. Under an additional decay hypothesis $u(x)=O(|x|^{2-n})$, the result was proved earlier by Obata [10] and Gidas, Ni and Nirenberg [5].

Our first result extends the above to all σ_k, $1\leqslant k\leqslant n$.

THEOREM 1. – *For $n\geqslant 3$ and $1\leqslant k\leqslant n$, let $u\in C^2(\mathbb{R}^n)$ be a positive solution of* (5) *satisfying* (6). *Then for some $a>0$ and $\bar{x}\in\mathbb{R}^n$,*

$$u(x)=c(n,k)\left(\frac{a}{1+a^2|x-\bar{x}|^2}\right)^{(n-2)/2}, \quad \forall x\in\mathbb{R}^n, \tag{8}$$

where $c(n,k)=2^{(n-2)/4}\binom{n}{k}^{(n-2)/4k}$.

Remark 1. – The case $k=2$ and $n=4$ was obtained by Chang, Gursky and Yang [4]. Under an additional hypothesis that $\frac{1}{|x|^{n-2}}u(x/(|x|^2))$ can be extended as a C^2 positive function near $x=0$, the case $2\leqslant k\leqslant n$ was obtained by Viaclovsky [16,17]. As mentioned above, the case $k=1$ was obtained by Caffarelli, Gidas and Spruck, while under an additional hypothesis that $\frac{1}{|x|^{n-2}}u(x/(|x|^2))$ is bounded near $x=0$, the case $k=1$ was obtained by Obata, and by Gidas, Ni and Nirenberg,

Remark 2. – By the ellipticity and the smoothness of $\sigma_k^{1/k}$ in Γ_k (*see* [2]), a C^2 solution of (5) satisfying (6) is in C^∞.

A crucial ingredient in our proof of Theorem 1 is the following Harnack type inequality.

THEOREM 2. – *For $n\geqslant 3$, $1\leqslant k\leqslant n$, and $R>0$, let $B_{3R}\subset\mathbb{R}^n$ be a ball of radius $3R$ and $u\in C^2(B_{3R})$ be a positive solution of*

$$\sigma_k\big(A^u\big)=1 \quad \textit{in } B_{3R}, \tag{9}$$

satisfying

$$\mu\big(A^u(x)\big)\in\Gamma_k, \quad \forall x\in B_{3R}. \tag{10}$$

Then

$$\Big(\max_{\overline{B}_R}u\Big)\Big(\min_{\overline{B}_{2R}}u\Big)\leqslant C(n)R^{2-n}. \tag{11}$$

Remark 3. – The above Harnack type inequality for $k=1$ was obtained by Schoen [13] based on the Liouville type theorem of Caffarelli, Gidas and Spruck. An important step toward our proof of Theorem 2 was taken in an earlier work of the second author and Zhang [9], where they gave a different proof of Schoen's Harnack type inequality which does not rely on the Liouville type theorem.

Based on Theorem 1, the classical interior $C^{2,\alpha}$ estimates of Evans and Krylov and the local estimates of Guan and Wang, we adapt the arguments of Schoen in [12] to obtain the following

THEOREM 3. – *For $n \geqslant 3$ and $1 \leqslant k \leqslant n$, let (M, g_0) be an n-dimensional smooth compact locally conformally flat Riemannian manifold without boundary satisfying*

$$\mu\big(A_{g_0}(x)\big) \in \Gamma_k, \quad \forall x \in M.$$

Then there exists some smooth positive function u on M such that $g = u^{4/(n-2)} g_0$ satisfies

$$\sigma_k\big(\mu(A_g)\big) = 1 \quad \text{in } M.$$

Moreover, if (M, g_0) is not conformally diffeomorphic to the standard n-sphere, all solutions of the above satisfy, for all $m \geqslant 0$, that

$$\|u\|_{C^m(M,g_0)} + \big\|u^{-1}\big\|_{C^m(M,g_0)} \leqslant C,$$

where C depends only on (M, g_0) and m.

Remark 4. – For $k = 1$, it is the Yamabe problem for locally conformally flat manifolds with positive Yamabe invariant, and the result is due to Schoen [11,12]. The Yamabe problem was solved through the work of Yamabe, Trudinger, Aubin, and Schoen. For $k = 2$ and $n = 4$, the result was proved without the locally conformally flatness hypothesis by Chang, Gursky and Yang [4].

Fully nonlinear elliptic equations of this type have been investigated in the classical paper of Caffarelli, Nirenberg and Spruck [2]. For extensive studies and outstanding results on such equations, see, e.g., Trudinger and Wang [14] and the references therein. Conformally invariant fully nonlinear equations (2) were introduced by Viaclovsky in [16]. Global gradient and second derivative estimates on general Riemannian manifolds for the equations were obtained by Viaclovsky in [15]. For similar equations on Riemannian manifolds of non-negative sectional curvature, global gradient and second derivative estimates were obtained by the second author in [8]. It remains open whether such estimates hold without any curvature hypothesis. Another question is, as raised to the second author by Ivochkina, whether the estimates hold for σ_k equations under some weaker σ_j curvature hypothesis. On 4-dimensional general Riemannian manifolds, remarkable results on (2) for $k = 2$ were obtained by Chang, Gursky and Yang in [3] and [4], which include Liouville type theorems, existence and compactness of solutions, as well as applications to topology. Guan and Wang recently extended in [6] the above mentioned global estimates of Viaclovsky to purely local estimates.

In the rest of this Note, we outline our proofs of Theorems 1 and 2. Our proof of Theorem 1 is completely different from the one for $k = 2$ and $n = 4$ by Chang, Gursky and Yang. First we adapt the method in [9] to establish Theorem 2. Next we establish Theorem 1 by the following procedure. Let u be the solution in Theorem 1. By the superharmonicity of u, we have

$$\liminf_{|x|\to\infty} \big(|x|^{n-2} u(x)\big) > 0. \tag{12}$$

Using Theorem 2 as a tool and with the help of the interior $C^{2,\alpha}$ estimates of Evans and Krylov and the local estimates in [6], we show that

$$\int_{\mathbb{R}^n} R_g u^{2n/(n-2)} < \infty, \tag{13}$$

and then we establish the stronger estimates

$$\limsup_{|x|\to\infty} \sum_{i=0}^{2} \big(|x|^{n-2+i} \big|\nabla^i u(x)\big|\big) < \infty, \tag{14}$$

where $g = u^{4/(n-2)} g_{\text{flat}}$, g_{flat} is the Euclidean metric, and R_g is the scalar curvature of g.

For $x \in \mathbb{R}^n$ and $\lambda > 0$, let $u_{x,\lambda}$ denote the Kelvin transformation of u:

$$u_{x,\lambda}(y) = \frac{\lambda^{n-2}}{|y-x|^{n-2}} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right), \quad y \in \mathbb{R}^n \setminus \{x\}.$$

By the conformal invariance of A_g [16], $u_{x,\lambda}$ satisfies the same equation as u. We then establish

LEMMA 1. – *For $n \geqslant 3$ and $1 \leqslant k \leqslant n$, let $w \in \mathrm{C}^2(\mathbb{R}^n \setminus \{0\})$ be a positive solution of*

$$\sigma_k\left(A^w\right) = 1 \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

satisfying

$$A^w(x) \in \Gamma_k, \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$

$w_{0,1}(x) \equiv (1/|x|^{n-2}) w(x/|x|^2)$ *can be extended as a positive function in* $\mathrm{C}^2(\mathbb{R}^n)$,

$$w \text{ can be extended as a continuous function in } \mathbb{R}^n, \tag{15}$$

and

$$\lim_{x \to 0} |x| \left|\nabla w(x)\right| = 0. \tag{16}$$

Then there exists some constant $0 < \lambda^* < \infty$ such that

$$w^{\lambda^*} \equiv w.$$

In particular, $w \in \mathrm{C}^2(\mathbb{R}^n)$.

Remark 5. – Hypotheses (15) can be dropped. In fact (15) and (16) can be replaced by a weaker hypothesis

$$\lim_{r \to 0} \mathrm{osc}_{\partial B_r} w = 0 \quad \text{and} \quad \lim_{r \to 0} \max_{\theta \in \mathbb{S}^{n-1}} \left(r \left|w_r(r,\theta)\right|\right) = 0.$$

By the superharmonicity of $w_{0,1}$, w is positive in $\mathbb{R}^n$. Lemma 1 is proved by first establishing, adapting the procedure in [9], that there exist $0 < \lambda_1 < \lambda_2 < \infty$ such that

$$w^\lambda(x) \geqslant w(x), \quad \forall 0 < |x| \leqslant \lambda \leqslant \lambda_1,$$

and

$$w^\lambda(x) \leqslant w(x), \quad \forall \lambda \geqslant \lambda_2,\ 0 < |x| \leqslant \lambda.$$

In the proof of one of the above, we make use of hypothesis (16).

Set

$$\underline{\lambda} = \sup\left\{\mu;\ w^\lambda(x) \geqslant w(x),\ \forall 0 < |x| \leqslant \lambda \leqslant \mu\right\},$$

and

$$\bar{\lambda} = \inf\left\{\mu;\ w^\lambda(x) \leqslant w(x),\ \forall \lambda \geqslant \mu,\ 0 < |x| \leqslant \lambda\right\}.$$

Obviously, $0 < \underline{\lambda} \leqslant \bar{\lambda} < \infty$. It is not difficult to see, by the strong maximum principle and the Hopf lemma, that $\underline{\lambda} = \bar{\lambda}$. Thus we have established the lemma.

We divide the rest of the proof into two cases. We start with the case $n/2 < k \leqslant n$. Recall that u is the solution in Theorem 1. Let

$$w(x) := u_{0,1}(x) \equiv \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right).$$

By (12) and (14), we have, for some constant $C \geqslant 1$, that

$$\frac{1}{C} \leqslant w(x) \leqslant C, \quad |x|\left|\nabla w(x)\right| + |x|^2\left|\nabla^2 w(x)\right| \leqslant C, \quad \forall x \in B_1 \setminus \{0\}. \tag{17}$$

We make the following simple but useful observation: let

$$\xi(x) = f\left(w(x)\right), \quad x \in B_1 \setminus \{0\},$$

where $f(s)=\frac{n-2}{2}s^{-2/(n-2)}$. Then

$$\frac{1}{|f'(w)|}\left(\nabla^2\xi\right)\geqslant\frac{n-2}{2}w^{(n+2)/(n-2)}A^w\quad\text{and therefore }\left(\nabla^2\xi\right)\in\Gamma_k\text{ in }B_1\backslash\{0\}.$$

For $k=n$, this implies that ξ is convex and $|\nabla\xi|$ is bounded in $B_{1/2}\backslash\{0\}$. Hence $|\nabla w|$ is bounded in $B_{1/2}\backslash\{0\}$ and (16) is satisfied. For $1\leqslant k\leqslant n$, let

$$\xi_\varepsilon(x)=\int\xi(x-y)\rho_\varepsilon(y)\,\mathrm{d}y,$$

where $\rho\in C_c^\infty(B_1)$ with $\rho\geqslant 0$ and $\int\rho=1$. By the property of ξ, we know that $\xi_\varepsilon\in C^\infty$, $D^2\xi_\varepsilon(x)\in\Gamma_k$ for $x\in B_{1/4}$ and $0<\varepsilon<1/4$. By Theorem 2.7 in [14], we have, for $\frac{n}{2}<k\leqslant n$, that $\|\xi_\varepsilon\|_{C^\alpha(B_{1/8})}$ is bounded by some constant depending only on k and n, where $\alpha=2-\frac{n}{k}$. It follows that $\xi\in C^\alpha(B_{1/8})$, i.e., $w\in C^\alpha(B_{1/8})$. By interpolation (*see* (17)), we have, for some positive constant C, that

$$\left|\nabla w(x)\right|\leqslant C|x|^{\alpha/2-1},\quad\forall x\in B_{1/9}\setminus\{0\}.$$

It follows that (16) is satisfied by w.

Applying Lemma 1 to w, we have $w\in C^2(\mathbb{R}^n)$. Theorem 1 in this case now follows from the result of Viaclovsky in the regular case.

The remaining case, $1\leqslant k\leqslant\frac{n}{2}$, is treated by a different argument. The proof is based on our strong decay estimates (14) together with the Obata type integral formula in [16].

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